

17B61, 17D30

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σ -Derivations

Quasi-Hom-
Lie algebras of
twisted vector
fields

Quasi-Lie,
Hom-Lie,
quasi-Hom-
Lie, color
Hom-Lie

Quasi-hom-Lie
algebras for
discretized
derivatives

Hom-
associative
and Hom-Lie
admissible
Hom-algebras

n -ary
Hom-Nambu,
Hom-Nambu-
Lie

Hom-algebra structures and n -ary Hom-algebra structures

AMS MSC 2020: 17B61 Hom-Lie and related algebras,

17D30 (non-Lie) Hom algebras and topics

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(Date of the talk: December 2, 2021)

- 1 σ -Derivations
- 2 Quasi-Hom-Lie algebras of twisted vector fields
- 3 Quasi-Lie, Hom-Lie, quasi-Hom-Lie, color Hom-Lie algebras
- 4 Quasi-hom-Lie algebras for discretized derivatives
- 5 Hom-associative and Hom-Lie admissible Hom-algebras
- 6 Hom-Nambu and Hom-Nambu-Lie n -ary algebras

- discretizations (quasi-deformations) of vector fields, (quantum) q -deformations of finite-dimensional Lie algebras and infinite-dimensional Lie algebras as Witt and Virasoro algebras; q -deformed vertex operator models of CFT; quantum field theory; quantization;
- q -deformed Heisenberg (Weyl) algebras, quantum oscillator algebras, quantum algebras, braided Lie algebras;
- q -analysis, q -special functions;
- q -deformations of differential and homological algebra, non-commutative twisted differential calculi and geometry;
- color Lie algebras and superalgebras (Γ -graded ϵ -Lie);
- quantum physics and quantum field theory;
- non-associative algebras;
- non-commutative geometry and non-commutative analysis;
- non-commutative probability and stochastic processes

σ -derivations

(twisted, deformed, discretized derivations)

\mathcal{A} – (commutative) associative \mathbb{K} -algebra with unity
 $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ algebra endomorphism

 σ -derivations

- $\partial_\sigma : \mathcal{A} \rightarrow \mathcal{A}$ linear map
- twisted (deformed) Leibniz rule

$$\partial_\sigma(a \cdot b) = \partial_\sigma(a) \cdot b + \sigma(a) \cdot \partial_\sigma(b)$$

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- the ordinary derivation operator

$$(\partial a)(x) = \lim_{y \rightarrow x} \frac{a(y) - a(x)}{y - x} = \frac{da}{dx}(x) = a'(x)$$

$$(\partial ab)(x) = (\partial a)(x)b(x) + a(x)(\partial b)(x)$$

$$\sigma = \text{id} : a(x) \mapsto a(x)$$

- Shifted difference operators

$$(\partial a)(x) = a(x + h) - a(x)$$

$$(\partial ab)(x) = (\partial a)(x)b(x) + a(x + h)(\partial b)(x)$$

$$\sigma(a)(x) = a(x + h)$$

- q -difference operator

$$(\partial a)(x) = a(qx) - a(x)$$

$$(\partial ab)(x) = (\partial a)(x)b(x) + a(qx)(\partial b)(x)$$

$$\sigma(a)(x) = a(qx)$$

- Jackson q -derivative

$$(\partial a)(x) = (D_q a)(x) = \frac{a(qx) - a(x)}{qx - x}$$

$$(\partial ab)(x) = (\partial a)(x)b(x) + a(qx)(\partial b)(x)$$

$$\sigma(a)(x) = a(qx)$$

$$\lim_{q \rightarrow 1} D_q(a)(x) = a'(x)$$

- **"General" σ -derivations (twisted difference operators)**

$\Omega \subset \mathbb{K}$ any subset of a field

$$T : \Omega \rightarrow \Omega$$

Any transformation without fixed points in Ω

A any algebra of functions a on Ω such that

$$\sigma(a)(x) = a(T(x)) \in A$$

$$\partial_\sigma : a(x) \mapsto \frac{a(T(x)) - a(x)}{T(x) - x} = \left(\frac{(\sigma - id)}{(T - id)} a \right) (x)$$

\Downarrow

$$\partial_\sigma(a \cdot b) = \partial_\sigma(a) \cdot b + \sigma(a) \cdot \partial_\sigma(b)$$

Theorem 1 \mathcal{A} is UFD (unique factorization domain)



Space of all σ -derivations $\mathfrak{D}_\sigma(\mathcal{A})$ is a free rank one \mathcal{A} -module with generator

$$\Delta = \frac{(\text{id} - \sigma)}{g} : a \mapsto \frac{(\text{id} - \sigma)(a)}{g}$$

where $g = \text{gcd}((\text{id} - \sigma)(\mathcal{A}))$

\mathcal{A} commutative algebra, $\sigma : \mathcal{A} \rightarrow \mathcal{A}$ algebra endomorphism, $\Delta : \mathcal{A} \rightarrow \mathcal{A}$ σ -derivation,

$Ann(\Delta) = \{a \in \mathcal{A} \mid a\Delta = 0\}$, σ -twisted vector fields $\mathcal{A} \cdot \Delta$

Theorem 2 (Hartwig, Larsson, Silvestrov, 2003)

J. of Algebra, 295 (2006), 314-361, Preprint Institute Mittag-Leffler, 2003, Preprint Lund University 2003

Bracket on $\mathcal{A} \cdot \Delta$ (well-defined if $\sigma(Ann(\Delta)) \subseteq Ann(\Delta)$)

$$\langle a \cdot \Delta, b \cdot \Delta \rangle_{\sigma} = (\sigma(a) \cdot \Delta)(b \cdot \Delta) - (\sigma(b) \cdot \Delta)(a \cdot \Delta)$$

Closure $\langle a \cdot \Delta, b \cdot \Delta \rangle_{\sigma} = (\sigma(a)\Delta(b) - \sigma(b)\Delta(a)) \cdot \Delta$

Skew-symmetry $\langle a \cdot \Delta, b \cdot \Delta \rangle_{\sigma} = -\langle b \cdot \Delta, a \cdot \Delta \rangle_{\sigma}$

Twisted 6 term Jacobi Identity

If $\Delta \circ \sigma(a) = \delta \cdot \sigma \circ \Delta(a)$, for some $\delta \in \mathcal{A}$, then $\forall a, b, c \in \mathcal{A}$:

$$\circlearrowleft_{a,b,c} \left(\langle \sigma(a) \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_{\sigma} \rangle_{\sigma} + \delta \cdot \langle a \cdot \Delta, \langle b \cdot \Delta, c \cdot \Delta \rangle_{\sigma} \rangle_{\sigma} \right) = 0$$

\mathcal{A} is UFD $\Rightarrow \delta = \frac{\sigma(g)}{g}$, $g = GCD(id - \sigma)(\mathcal{A})$

Quasi-Lie algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

$$(L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega, \theta)$$

1. L is a linear space over \mathbb{F} ,
2. $\langle \cdot, \cdot \rangle_L : L \times L \rightarrow L$ is a bilinear product or bracket in L ;
3. $\alpha, \beta : L \rightarrow L$, are linear maps,
4. $\omega : D_\omega \rightarrow \mathcal{L}_{\mathbb{F}}(L)$ and $\theta : D_\theta \rightarrow \mathcal{L}_{\mathbb{F}}(L)$ are maps with domains of definition $D_\omega, D_\theta \subseteq L \times L$,

ω -Symmetry $\forall (x, y) \in D_\omega$

$$\langle x, y \rangle_L = \omega(x, y) \langle y, x \rangle_L,$$

Quasi-Jacobi identity $\forall (z, x), (x, y), (y, z) \in D_\theta$

$$\circlearrowleft_{x,y,z} \{ \theta(z, x) (\langle \alpha(x), \langle y, z \rangle_L \rangle_L + \beta \langle x, \langle y, z \rangle_L \rangle_L) \} = 0$$

Hom-Lie algebras is special subclass of Quasi-Lie algebras

$$\beta = 0, \quad \omega = -\text{id}_L, \quad \theta = \text{id}_L$$

1. linear map $\alpha : L \rightarrow L$
2. bilinear multiplication (bracket) $\langle \cdot, \cdot \rangle_\alpha$ such that
 - **skew-symmetry** $\langle x, y \rangle_\alpha = -\langle y, x \rangle_\alpha$
 - **Hom-Lie Jacobi identity** $\forall x, y, z \in L$

$$\circlearrowleft_{x,y,z} \langle \alpha(x), \langle y, z \rangle_\alpha \rangle_\alpha = 0$$

$(L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega)$

1. L is a linear space over field \mathbb{K}
2. $\langle \cdot, \cdot \rangle_L : L \times L \rightarrow L$ is a bilinear map
3. $\alpha, \beta : L \rightarrow L$ are linear maps
4. $\omega : D_\omega \rightarrow \text{End}(L)$ is a map with domain of definition $D_\omega \subseteq L \times L$ taking values in linear operators on L
 - (**β -twisting**) α is a β -twisted algebra endomorphism:

$$\langle \alpha(x), \alpha(y) \rangle_L = \beta \circ \alpha \langle x, y \rangle_L \quad \forall x, y \in L$$

- (**ω -symmetry**) $\langle x, y \rangle_L = \omega(x, y) \langle y, x \rangle_L \quad \forall (x, y) \in D_\omega$
- **Quasi-Hom-Lie Jacobi identity**
 $\forall (z, x), (x, y), (y, z) \in D_\omega$

$$\mathcal{O}_{x,y,z} \left\{ \omega(z, x) \left(\langle \alpha(x), \langle y, z \rangle_L \rangle_L + \beta \langle x, \langle y, z \rangle_L \rangle_L \right) \right\} = 0$$

First introduced in:

D. Larsson, S. Silvestrov, Graded quasi-Lie algebras, Czechoslovak J. Phys., 55, 11 (2005), 1473-1478

 Γ -graded (color) quasi-Lie algebra $(L, \langle \cdot, \cdot \rangle_L, \alpha, \beta, \omega, \theta)$
 $(\Gamma, \hat{+})$ commutative semigroup

1. $L = \bigoplus_{\gamma \in \Gamma} L_\gamma$ is a Γ -graded linear space over \mathbb{F} ,
2. $\langle \cdot, \cdot \rangle_L : L \times L \rightarrow L$ is a bilinear map (bracket);
3. $\alpha, \beta : L \rightarrow L$ are linear maps,

$$\alpha(\cup_{\gamma \in \Gamma} L_\gamma) \subseteq \cup_{\gamma \in \Gamma} L_\gamma, \quad \beta(\cup_{\gamma \in \Gamma} L_\gamma) \subseteq \cup_{\gamma \in \Gamma} L_\gamma$$

$$D_\omega, D_\theta \subseteq \cup_{\gamma \in \Gamma} L_\gamma \times \cup_{\gamma \in \Gamma} L_\gamma$$

$$\omega : D_\omega \rightarrow \mathcal{L}_{\mathbb{F}}(L), \quad \theta : D_\theta \rightarrow \mathcal{L}_{\mathbb{F}}(L)$$

 Γ -grading axiom $\forall \gamma_1, \gamma_2 \in \Gamma : \langle L_{\gamma_1}, L_{\gamma_2} \rangle_L \subseteq L_{\gamma_1 \hat{+} \gamma_2}$ ω -symmetry $\forall (x, y) \in D_\omega : \langle x, y \rangle_L = \omega(x, y) \langle y, x \rangle_L$ quasi-Jacobi identity $\forall (z, x) \in D_\theta, (x, y) \in D_\theta, (y, z) \in D_\theta$

$$\bigcirc_{x, y, z} \{ \theta(z, x) (\langle \alpha(x), \langle y, z \rangle_L \rangle_L + \beta \langle x, \langle y, z \rangle_L \rangle_L) \} = 0$$

Note: $(\omega(x, y)\omega(y, x) - \text{id})\langle x, y \rangle = 0$, if $(x, y), (y, x) \in D_\omega$ by ω -symmetry

Hom-Lie color algebras is special subclass of color (graded) quasi-Lie algebras first introduced in:

D. Larsson, S. Silvestrov, Graded quasi-Lie algebras, Czechoslovak J. Phys., 55, 11 (2005), 1473-1478

Hom-Lie superalgebras $\Gamma = \mathbb{Z}_2$ and $\varepsilon(a, b) = (-1)^{|a||b|}$

$(L, [\cdot, \cdot], \alpha, \varepsilon)$

L is Γ -graded space

$[\cdot, \cdot] : L \times L \rightarrow L$ is an even bilinear mapping

α is an even linear map

ε is bi-character on Γ

ε -skew-symmetry $[x, y] = -\varepsilon(x, y)[y, x]$,

Hom ε -Jacobi identity

$\bigcirc_{x,y,z} \varepsilon(z, x)[\alpha(x), [y, z]] = 0$,

for all homogenous elements x, y, z in L .

Γ – commutative group (or semigroup). K field of $\text{char} K \neq 2, 3$ Γ -graded algebra

$$L = \bigoplus_{\gamma \in \Gamma} L_{\gamma}$$

 $\langle \cdot, \cdot \rangle : L \times L \longrightarrow L$ bilinear map

$$\forall A \in L_{\alpha}, B \in L_{\beta}, C \in L_{\gamma}, \quad \alpha, \beta, \gamma \in \Gamma:$$

$$\langle A, B \rangle = -\varepsilon(\alpha, \beta) \langle B, A \rangle \quad (\varepsilon\text{-skew symmetry})$$

$$\varepsilon(\gamma, \alpha) \langle A, \langle B, C \rangle \rangle + \varepsilon(\beta, \gamma) \langle C, \langle A, B \rangle \rangle + \varepsilon(\alpha, \beta) \langle B, \langle C, A \rangle \rangle = 0$$

(ε -Jacoby identity)

Color Lie algebras are examples of quasi Hom-Lie algebras.

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n -ary Hom-Nambu, Hom-Nambu-Lie

L Γ -graded quasi Hom-Lie algebra (color Lie algebra)

$$L = \bigoplus_{\gamma \in \Gamma} L_{\gamma}$$

$$\alpha = \beta = \text{id}_L, \quad \omega(x, y)v = -\varepsilon(\gamma_x, \gamma_y)v$$

$$v \in L \text{ and } (x, y) \in D_{\omega} = \bigcup_{\gamma \in \Gamma} L_{\gamma}$$

$\gamma_x, \gamma_y \in \Gamma$ graded degrees of x and y .

The ω -symmetry and the quasi-hom-Lie-Jacobi identity



Γ -Graded ε -symmetry and ε -Jacobi identities for color Lie algebras.

Lie superalgebras

$$\Gamma = \mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z},$$

$$\varepsilon(\gamma_x, \gamma_y) = (-1)^{\gamma_x \gamma_y}$$

$\gamma_x \gamma_y$ is the product in \mathbb{Z}_2 .

Quasi-Leibniz algebras were first introduced in

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Quasi-Leibniz-Loday algebra

For $(z, x), (x, y), (y, z) \in D_\theta, (\alpha(z), \langle x, y \rangle), (\alpha(y), \langle z, x \rangle), (y, \langle z, x \rangle), (z, x) \in D_\omega$

$$\begin{aligned} \theta(y, z)(\omega(\alpha(z), \langle x, y \rangle)\langle \langle x, y \rangle, \alpha(z) \rangle + \beta \circ \omega(z, \langle x, y \rangle)\langle \langle x, y \rangle, z \rangle) = \\ = -\theta(x, y)(\omega(\alpha(y), \langle z, x \rangle)\langle \omega(z, x)\langle x, z \rangle, \alpha(y) \rangle + \\ + \beta \circ \omega(y, \langle z, x \rangle)\langle \omega(z, x)\langle x, z \rangle, y \rangle) - \\ - \theta(z, x)(\langle \alpha(x), \langle y, z \rangle \rangle + \beta\langle x, \langle y, z \rangle \rangle) \end{aligned}$$

For $\alpha = \text{id}, \beta = 0$ and $\theta = \omega = -\text{id}$, **Leibniz-Loday algebra**
 $\langle \langle x, y \rangle, z \rangle = \langle \langle x, z \rangle, y \rangle + \langle x, \langle y, z \rangle \rangle.$

Hom-Leibniz algebras (Hom-Leibniz-Loday algebras)

(special subclass of quasi-Leibniz algebras)

Quasi-Leibniz algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

Hom-Leibniz-Loday algebra is a triple $(V, \langle \cdot, \cdot \rangle, \alpha)$, where V is a linear space,

$\alpha : V \rightarrow V$ is a linear map (linear space endomorphism of V)

$\langle \cdot, \cdot \rangle : V \times V \rightarrow V$ is a bilinear map

satisfying:

$$\langle \langle x, y \rangle, \alpha(z) \rangle = \langle \langle x, z \rangle, \alpha(y) \rangle + \langle \alpha(x), \langle y, z \rangle \rangle$$

Hom-Lie algebras are skew-symmetric Hom-Leibniz algebras

$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot d_n$$

$$\Delta = tD_q = \frac{\sigma - \text{id}}{q-1} : f(t) \mapsto \frac{f(qt) - f(t)}{q-1}$$

$$\sigma(t) = qt, \quad \sigma(f)(t) = f(qt), \quad \{n\}_q = \frac{q^n - 1}{q - 1}$$

Skew-symmetric product. $d_n = -t^n \Delta$

$$\langle d_n, d_m \rangle = q^n d_n d_m - q^m d_m d_n = (\{n\}_q - \{m\}_q) d_{n+m}$$

Graded Hom-Lie algebra $\langle L_n, L_m \rangle \subseteq L_{n+m}$

Hom-Lie algebra Jacobi-identity

$$\circlearrowleft_{n,m,l} (q^n + 1) \langle d_n, \langle d_m, d_l \rangle \rangle = 0$$

$$\alpha(d_n) = (q^n + 1) d_n$$

$$\circlearrowleft_{n,m,l} \langle \alpha(d_n), \langle d_m, d_l \rangle \rangle = 0$$

$$(\mathrm{Vir}_q, \hat{\sigma}) = (\mathrm{Witt}_q \oplus \mathbb{C} \cdot c, \hat{\sigma}) \quad \{d_n : n \in \mathbb{Z}\} \cup \{c\}$$

$$\hat{\sigma} : \mathrm{Vir}_q \rightarrow \mathrm{Vir}_q, \quad \hat{\sigma}(d_n) = q^n d_n, \quad \hat{\sigma}(c) = c$$

$$\langle d_n, d_m \rangle = (\{n\}_q - \{m\}_q) d_{n+m} +$$

$$+ \delta_{n+m,0} \frac{q^{-n}}{6(1+q^n)} \{n-1\}_q \{n\}_q \{n+1\}_q c$$

$$\langle c, \mathrm{Vir}_q \rangle = 0$$

Quasi-Hom-Lie algebra \mathfrak{g}



Linear space

$$\hat{\mathfrak{g}} := \mathfrak{g} \otimes \mathbb{K}[t, t^{-1}]$$

The algebra of Laurent polynomials with coefficients in the qhl-algebra \mathfrak{g} .

$$\alpha_{\hat{\mathfrak{g}}} := \alpha_{\mathfrak{g}} \otimes \text{id}$$

$$\beta_{\hat{\mathfrak{g}}} := \beta_{\mathfrak{g}} \otimes \text{id}$$

$$\omega_{\hat{\mathfrak{g}}} := \omega_{\mathfrak{g}} \otimes \text{id}$$

$$\langle x \otimes t^n, y \otimes t^m \rangle_{\hat{\mathfrak{g}}} = \langle x, y \rangle_{\mathfrak{g}} \otimes t^{n+m}$$

$\hat{\mathfrak{g}}$ is a quasi Hom-Lie algebra.

Non-linear Quasi-Lie deformations of Witt algebra

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$$\mathfrak{D}_\sigma(\mathcal{A}) = \bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot d_n,$$

$$D = \alpha t^{-k+1} \frac{\text{id} - \sigma}{t - qt^s}, \quad \sigma(t) = qt^s$$

Skew-symmetric product $d_n = -t^n D$

$$\langle d_n, d_m \rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n$$

$$\langle d_n, d_m \rangle_\sigma = \text{linear combinations of generators}$$

For $n, m \geq 0$:

$$\langle d_n, d_m \rangle_\sigma = \alpha \text{sign}(n - m) \sum_{l=\min(n,m)}^{\max(n,m)-1} q^{n+m-1-l} d_{s(n+m-1)-(k-1)-l(s-1)}$$

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σ -deformed Jacobi-identity

$$\circlearrowleft_{n,m,l} \left(\langle q^n d_{ns}, \langle d_m, d_l \rangle_\sigma \rangle_\sigma + \underbrace{q^k t^{k(s-1)} \sum_{r=0}^{s-1} (qt^{s-1})^r \langle d_n \langle d_m, d_l \rangle_\sigma \rangle_\sigma}_{=\delta} \right) = 0.$$

Quasi-Hom-Lie algebra, not Hom-Lie algebra for $s \neq 1$

Other non-linear Quasi-Lie deformations of Witt algebra

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$$\sigma(t) = qt^s$$

$$D = \frac{\text{id} - \sigma}{\eta^{-1} \cdot t^k}$$

generates a cyclic \mathcal{A} -submodule \mathfrak{M} of $\mathfrak{D}_\sigma(\mathcal{A})$, proper for $s \neq 1$ ($s \neq 1$: $\sigma(t) = \beta t$ for some $\beta \in \mathbb{K}$)

Theorem The linear space

$$\mathfrak{M} = \bigoplus_{i \in \mathbb{Z}} \mathbb{K} \cdot d_i \quad \text{with} \quad d_i = -t^i D$$

is a quasi-Lie algebra

$$\langle d_n, d_m \rangle_\sigma = q^n d_{ns} d_m - q^m d_{ms} d_n = \eta q^m d_{ms+n-k} - \eta q^n d_{ns+m-k}$$

$$s \in \mathbb{Z} \text{ and } \eta \in \mathbb{C}$$

The σ -deformed Jacobi identity

$$\circlearrowleft_{n,m,l} \left(\langle \langle q^n d_{ns}, \langle d_m, d_l \rangle_\sigma \rangle_\sigma + \underbrace{q^k t^{(s-1)k}}_{=\delta} \langle d_n, \langle d_m, d_l \rangle_\sigma \rangle_\sigma \right) = 0$$

$q = 1$, $k = 0$ and $s = 1$

get a commutative algebra with countable number of generators instead of the Witt algebra.

Non-linear Quasi-Lie deformations of Witt algebra are almost graded

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Almost graded algebras $s = 1$:

$$\langle L_n, L_m \rangle_\sigma \subseteq L_{n+m-k}$$

(quasi-Lie deformations of) Krichever-Novikov type algebras,

Graded $k = 0, s = 1$: $\langle L_n, L_m \rangle_\sigma \subseteq L_{n+m}$

Hyper almost Graded algebras:

$$\langle L_n, L_m \rangle_\sigma \subseteq \bigoplus_{j \in \mathbb{Z} \cap [ms+n-k, ns+m-k]} L_j$$

$$ms + n - k = m + n + m(s - 1) - k$$

Hom-associative algebras \mapsto Hom-Lie algebras

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Hom-associative algebras were first introduced in

Makhlouf A., Silvestrov S.D., Hom-algebra structures, J. Gen. Lie Theory, Appl. 2 (2), 51–64 (2008).

(Preprints in Mathematical Sciences 2006:10, LUTFMA-5074-2006, Centre for Mathematical Sciences,

Department of Mathematics, Lund Institute of Technology, Lund University, 2006)

Hom-associative algebra (V, μ, α)

V linear space,

$\mu : V \times V \rightarrow V$ bilinear map,

$\alpha : V \rightarrow V$ linear map (linear space homomorphism)

Hom-associativity (two notations for multiplication)

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$$

$$\alpha(x)(yz) = (xy)\alpha(z)$$

$\alpha = Id_V \Leftrightarrow$ associative algebra

Theorem

Hom-associative algebras are **Hom-Lie admissible**.

(V, μ, α) is Hom-associative algebra $\alpha(x)(yz) = (xy)\alpha(z)$

$$\Downarrow \quad [\cdot, \cdot]_{\alpha} = xy - yx$$

$(V, [\cdot, \cdot]_{\alpha}, \alpha)$ is Hom-Lie algebra

- **skew-symmetry** $[x, y]_{\alpha} = -[y, x]_{\alpha}$
- **Hom-Lie Jacobi identity** $\forall x, y, z \in L$

$$\circlearrowleft_{(x,y,z)} [\alpha(x), [y, z]_{\alpha}]_{\alpha} =$$

$$[\alpha(x), [y, z]_{\alpha}]_{\alpha} + [\alpha(z), [x, y]_{\alpha}]_{\alpha} + [\alpha(y), [z, x]_{\alpha}]_{\alpha} = 0$$

G -Hom-associative algebras \mapsto Hom-Lie algebras

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$G \subseteq \mathcal{S}_3$ subgroup of the permutations group

$(-1)^{\varepsilon(s)}$ is the signature of the permutation

$$s(x_1, x_2, x_3) = (x_{s(1)}, x_{s(2)}, x_{s(3)})$$

G -Hom-associative Hom-algebra (V, μ, α)

$$\sum_{s \in G \subseteq \mathcal{S}_3} (-1)^{\varepsilon(s)} \underbrace{(\mu(\mu(x_{s(1)}, x_{s(2)}), \alpha(x_{s(3)})) - \mu(\alpha(x_{s(1)}), \mu(x_{s(2)}, x_{s(3)})))}_{a_{\mu, \alpha}} =$$

$$\sum_{s \in G \subseteq \mathcal{S}_3} (-1)^{\varepsilon(s)} \underbrace{((x_{s(1)}x_{s(2)})\alpha(x_{s(3)}) - \alpha(x_{s(1)})(x_{s(2)}x_{s(3)}))}_{a_{\alpha}} = 0,$$

\Leftrightarrow

$$\sum_{s \in G} (-1)^{\varepsilon(s)} a_{\mu, \alpha} \circ \sigma = 0$$

Hom-associator (α -associator): $a_{\alpha}(x, y, z) = (xy)\alpha(z) - \alpha(x)(yz)$

Theorem

G-Hom-associative algebras are Hom-Lie admissible:

For any G-Hom-associative algebra (V, μ, α) ,

$(V, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra

with commutator multiplication $[x, y] = \mu(x, y) - \mu(y, x)$

- **skew-symmetry** $[x, y]_\alpha = -[y, x]_\alpha$
- **Hom-Lie Jacobi identity** $\forall x, y, z \in L$

$$\circlearrowleft_{(x,y,z)} [\alpha(x), [y, z]_\alpha]_\alpha =$$

$$[\alpha(x), [y, z]_\alpha]_\alpha + [\alpha(z), [x, y]_\alpha]_\alpha + [\alpha(y), [z, x]_\alpha]_\alpha = 0$$

The subgroups of \mathcal{S}_3 are

$$G_1 = \{Id\}, \quad G_2 = \{Id, \tau_{12}\}, \quad G_3 = \{Id, \tau_{23}\}$$

$$G_4 = \{Id, \tau_{13}\}, \quad G_5 = A_3, \quad G_6 = \mathcal{S}_3$$

A_3 is the alternating group;

τ_{ij} is the transposition of i and j .

G -Hom-associative algebras \mapsto Hom-Lie algebras

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n -ary Hom-Nambu, Hom-Nambu-Lie

- The G_1 -Hom-associative algebras are the Hom-associative algebras.
- The G_2 -Hom-associative algebras satisfy

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(y), \mu(x, z)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(y, x), \alpha(z))$$

Vinberg algebra or left symmetric algebra: $\alpha = Id$

- The G_3 -Hom-associative algebras satisfy

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(x), \mu(z, y)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(x, z), \alpha(y))$$

Hom-pre-Lie algebra

$$\alpha(x)(yz) - \alpha(x)(zy) = (xy)\alpha(z) - (xz)\alpha(y)$$

Pre-Lie algebra or right symmetric algebra: $\alpha = Id$

G -Hom-associative algebras \mapsto Hom-Lie algebras

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- The G_4 -Hom-associative algebras satisfy

$$\mu(\alpha(x), \mu(y, z)) - \mu(\alpha(z), \mu(y, x)) = \mu(\mu(x, y), \alpha(z)) - \mu(\mu(z, y), \alpha(x))$$

- The G_5 -Hom-associative algebras satisfy the condition

$$\mu(\alpha(x), \mu(y, z)) + \mu(\alpha(y), \mu(z, x)) + \mu(\alpha(z), \mu(x, y)) = \mu(\mu(x, y), \alpha(z)) + \mu(\mu(y, z), \alpha(x)) + \mu(\mu(z, x), \alpha(y))$$

Note: μ skew-symmetric \Rightarrow the Hom-Jacobi identity.

- The G_6 -Hom-associative algebras are the Hom-Lie admissible algebras.

From Lie algebras to Hom-Lie algebras. Composition trick

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$(V, [\cdot, \cdot])$ Lie algebra

$\alpha : V \rightarrow V$ Lie algebra endomorphism

$$[x, y]_{\alpha} = \alpha([x, y])$$

Then $(V, [\cdot, \cdot]_{\alpha})$ is a Hom-Lie algebra

$$[x, y]_{\alpha} = -[y, x]_{\alpha}, \quad \circlearrowleft_{x, y, z} [[\alpha(x), [y, z]_{\alpha}]_{\alpha} = 0.$$

Hom-Leibniz algebras (Hom-Loday algebras)

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Special subclass of quasi-Leibniz algebras

Quasi-Leibniz algebras were first introduced in

D. Larsson, S. Silvestrov, Quasi-Lie algebras, In: Jurgen Fuchs, et al. (eds), "Noncommutative Geometry and Representation Theory in Mathematical Physics", American Mathematical Society, Contemporary Mathematics, Vol. 391, 2005.

Definition $(V, \langle \cdot, \cdot \rangle, \alpha)$ consisting of a linear space V , bilinear map $\langle \cdot, \cdot \rangle : V \times V \rightarrow V$ and a homomorphism $\alpha : V \rightarrow V$ satisfying

$$\langle \langle x, y \rangle, \alpha(z) \rangle = \langle \langle x, z \rangle, \alpha(y) \rangle + \langle \alpha(x), \langle y, z \rangle \rangle.$$

If a Hom-Leibniz algebra is skewsymmetric then it is a Hom-Lie algebra.

Hom-Poisson algebra

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Makhlouf A., Silvestrov S.D., Notes on 1-Parameter Formal Deformations of Hom-associative and Hom-Lie Algebras, Forum Math. 22 (4), 715-739 (2010).

(Preprints in Mathematical Sciences, Lund University, Centre for Mathematical Sciences, Centrum Scientiarum Mathematicarum, (2007:31) LUTFMA-5095-2007 (2007); arXiv:0712.3130 (2007)).

Definition

$(V, \mu, \{\cdot, \cdot\}, \alpha)$, V linear space, $\mu : V \times V \rightarrow V$ and $\{\cdot, \cdot\} : V \times V \rightarrow V$ bilinear maps, $\alpha : V \rightarrow V$ linear map:

- 1) (V, μ, α) is a commutative Hom-associative algebra
- 2) $(V, \{\cdot, \cdot\}, \alpha)$ is a Hom-Lie algebra
- 3) for all x, y, z in V ,

$$\{\alpha(x), \mu(y, z)\} = \mu(\alpha(y), \{x, z\}) + \mu(\alpha(z), \{x, y\}).$$

Equivalently, for all x, y, z in V , $ad_z(\cdot) = \{\cdot, z\}$ is a Hom-derivation for the multiplication μ :

$$\{\mu(x, y), \alpha(z)\} = \mu(\{x, z\}, \alpha(y)) + \mu(\alpha(x), \{y, z\})$$

Let $\mathcal{A}_t = (V, \mu_t, \alpha_t)$ be a deformation of the commutative Hom-associative algebra

$$\mathcal{A}_0 = (V, \mu_0, \alpha_0)$$

$$\mu_t(x, y) = \mu_0(x, y) + \mu_1(x, y)t + \mu_2(x, y)t^2 + \dots$$

Then

$$\frac{\mu_t(x, y) - \mu_t(y, x)}{t} = \mu_1(x, y) - \mu_1(y, x) + t \sum_{i \geq 2} (\mu_i(x, y) - \mu_i(y, x)) t^{i-2}$$

Hence, if t goes to zero then $\frac{\mu_t(x, y) - \mu_t(y, x)}{t}$ goes to $\{x, y\} := \mu_1(x, y) - \mu_1(y, x)$

Theorem

$$\mathcal{A}_0 = (V, \mu_0, \alpha_0)$$

a commutative Hom-associative algebra

$$\mathcal{A}_t = (V, \mu_t, \alpha_t) \text{ a deformation of } \mathcal{A}_0.$$

Consider the bracket

$$\{x, y\} = \mu_1(x, y) - \mu_1(y, x)$$

is the first order element of the deformation μ_t .



$(V, \mu_0, \{, \}, \alpha_0)$ is a Hom-Poisson algebra.

Hom-Nambu and Hom-Nambu-Lie n -ary algebras

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n -ary Hom-Nambu, Hom-Nambu-Lie

H. Ataguema, A. Makhlouf, S. Silvestrov, Generalization of n -ary Nambu Algebras and Beyond, Journal of Mathematical Physics, 50, 083501,2009

Definition

An n -ary Hom-Nambu algebra is a triple $(V, [\cdot, \dots, \cdot], \alpha)$, where V is linear space,

$\alpha = (\alpha_j)_{j=1, \dots, n-1}$ is a family of linear maps $\alpha_j : V \rightarrow V$,
 $[\cdot, \dots, \cdot] : V^{\times n} \rightarrow V$ is n -linear map (n -ary product) satisfying:

The n -ary Hom-Nambu identity

$$[\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), [x_n, \dots, x_{2n-1}]] = \sum_{i=n}^{2n-1} [\alpha_1(x_n), \dots, \alpha_{i-n}(x_{i-1}), [x_1, \dots, x_{n-1}, x_i], \alpha_{i-n+1}(x_{i+1}), \dots, \alpha_{n-1}(x_{2n-1})]$$

for all $(x_1, \dots, x_{2n-1}) \in V^{2n-1}$.

Ternary Hom-Nambu algebras

$$\begin{aligned} [\alpha_1(x_1), \alpha_2(x_2), [x_3, x_4, x_5]] = \\ [[x_1, x_2, x_3], \alpha_1(x_4), \alpha_2(x_5)] + [\alpha_1(x_3), [x_1, x_2, x_4], \alpha_2(x_5)] \\ + [\alpha_1(x_3), \alpha_2(x_4), [x_1, x_2, x_5]]. \end{aligned}$$

n -ary Hom-Nambu algebras

Theorem. Let (V, m) be an n -ary Nambu algebra and let $\rho : V \rightarrow V$ be an n -ary Nambu algebras endomorphism.

$$m_\rho = \rho \circ m$$

$$\tilde{\rho} = (\rho, \dots, \rho).$$

Then $(V, m_\rho, \tilde{\rho})$ is an n -ary Hom-Nambu algebra.

Definition

A ternary Hom-Nambu algebra $(V, [\cdot, \cdot, \cdot], (\alpha_1, \alpha_2))$ is called a *ternary Hom-Nambu-Lie algebra* if the bracket is skew-symmetric, that is

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}] = \text{Sgn}(\sigma)[x_1, x_2, x_3]$$

$$\forall \sigma \in \mathcal{S}_3 \text{ and } \forall x_1, x_2, x_3 \in V$$

J. Arnlind, A. Makhlouf, S. Silvestrov, Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras, J. Mathematical Physics 51, 1, 2010

Definition

$(V, [\cdot, \cdot])$ binary algebra

$\tau : V \rightarrow \mathbb{K}$ linear map.

Define ternary bracket (trilinear map)

$[\cdot, \cdot, \cdot]_{\tau} : V \times V \times V \rightarrow V$:

$$[x, y, z]_{\tau} = \tau(x)[y, z] + \tau(y)[z, x] + \tau(z)[x, y].$$

Hom-Nambu-Lie algebras induced from Hom-Lie algebras

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If the bilinear multiplication $[\cdot, \cdot]$ in Definition 3 is skew-symmetric, then the trilinear map $[\cdot, \cdot, \cdot]_{\mathcal{T}}$ is skew-symmetric as well.

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If τ is a linear function such that $\tau([x, y]) = 0$ for all $x, y \in V$, then we call τ a *trace function on* $(V, [\cdot, \cdot])$. It follows immediately that $\tau([x, y, z]_{\tau}) = 0$ for all $x, y, z \in V$ if τ is a trace function.

Theorem

$(V, [\cdot, \cdot], \alpha)$ be a Hom-Lie algebra and $\beta : V \rightarrow \mathbb{K}$ be a linear map. Assume that τ is a trace function on V fulfilling

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y))$$

$$\tau(\beta(x))\tau(y) = \tau(x)\tau(\beta(y))$$

$$\tau(\alpha(x))\beta(y) = \tau(\beta(x))\alpha(y)$$

for all $x, y \in V$.

Then $(V, [\cdot, \cdot, \cdot]_{\tau}, (\alpha, \beta))$ is a Hom-Nambu-Lie algebra.

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If we choose $\beta = \alpha$ conditions for trace τ reduce to

$$\tau(\alpha(x))\tau(y) = \tau(x)\tau(\alpha(y)).$$

Example

V vector space of $n \times n$ matrices

$\alpha(x) = s^{-1}xs$ for an invertible matrix s

Then $(V, \alpha \circ [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra. For matrices, any trace function is proportional to the matrix trace, so we let $\tau(x) = \text{tr}(x)$. If we want to choose a $\beta \neq 0$, it can be proved that β has to be proportional to α , i.e. $\beta = \lambda\alpha$ for some $\lambda \neq 0$. Since $\text{tr}(\alpha(x)) = \text{tr}(x)$ it is clear that $(\alpha, \lambda\alpha, \text{tr})$ is a nondegenerate compatible triple on V , which implies that $(V, [\cdot, \cdot, \cdot]_{\text{tr}}, (\alpha, \lambda\alpha))$ is a Hom-Nambu-Lie algebra induced from $(V, \alpha \circ [\cdot, \cdot], \alpha)$.

Example

Let us start with the vector space V spanned by $\{x_1, x_2, x_3, x_4\}$ with a skew-symmetric bilinear map defined through

$$[x_i, x_j] = a_{ij}x_3 + b_{ij}x_4$$

where a_{ij} and b_{ij} are antisymmetric 4×4 matrices. Defining

$$\begin{aligned} \alpha(x_i) &= x_3 & \beta(x_i) &= x_4 & i &= 1, \dots, 4 \\ \tau(x_1) &= \gamma_1 & \tau(x_2) &= \gamma_2 & \tau(x_3) &= \tau(x_4) = 0, \end{aligned}$$

one immediately observes that τ is a trace function, $\text{im } \alpha \subseteq \ker \tau$, $\text{im } \beta \subseteq \ker \tau$, and $\beta \neq \alpha$.

Example cont.

Furthermore, $(V, [\cdot, \cdot], \alpha)$ is a Hom-Lie algebra provided

$$b_{13} = b_{12} + b_{23}$$

$$b_{14} = b_{12} + b_{23} + b_{34}$$

$$b_{24} = b_{23} + b_{34}.$$

Example cont.

By introducing $a = b_{12}$, $b = b_{23}$ and $c = b_{34}$, the four independent ternary brackets of the induced Hom-Nambu-Lie algebra can be written as

$$[x_1, x_2, x_3] = (\gamma_1 a_{23} - \gamma_2 a_{13})x_3 + (\gamma_1 b - \gamma_2(a + b))x_4$$

$$[x_1, x_2, x_4] = (\gamma_1 a_{24} - \gamma_2 a_{14})x_3 + (\gamma_1(b + c) - \gamma_2(a + b + c))x_4$$

$$[x_1, x_3, x_4] = (\gamma_1 a_{34})x_3 + (\gamma_1 c)x_4$$

$$[x_2, x_3, x_4] = (\gamma_2 a_{34})x_3 + (\gamma_2 c)x_4.$$

Example cont.

For instance, choosing $\gamma_1 = \gamma_2 = 1$ and $a_{i < j} = 1$, one obtains the Hom-Nambu-Lie algebra

$$(\langle x_1, x_2, x_3, x_4 \rangle, [\cdot, \cdot, \cdot], (\alpha, \beta))$$

defined by

$$[x_1, x_2, x_3] = -ax_4$$

$$[x_1, x_2, x_4] = -cx_4$$

$$[x_1, x_3, x_4] = x_3 + cx_4$$

$$[x_2, x_3, x_4] = x_3 + cx_4$$

together with $\alpha(x_i) = x_3$ and $\beta(x_i) = x_4$.

Some key references on Hom-algebra structures

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Some key references on Hom-algebra structures (cont.)

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Thank you for the music,
the songs I'm singing!
Thanks for all the joy
they're bringing!