# Strong hom-associativity II′

Lars Hellström
~~MDH~~ MDU

~~30 September 2019, SPAS, Västerås~~
2 December 2021, SNAG, online

## Abstract

Strong hom-associativity is a mild strengthening of ordinary hom-associativity, which leads to a more straightforward rewrite theory. Most examples of hom-associative algebras that have been examined turn out to be strongly hom-associative.

# Strong hom-associativity

The *strong* variant of hom-associativity generalises the identity

$$\mu\bigl(\mu(x,y),\alpha(z)\bigr) = \mu\bigl(\alpha(x),\mu(y,z)\bigr)$$

to allow a $\mu$–$\alpha$ exchange at an arbitrary nesting height above a multiplication $\mu$:

$$\mu\Bigl(\tfrac{\mu}{\alpha}\bigl(\cdots\mu(x,y)\cdots\bigr),\tfrac{\mu}{\alpha}\bigl(\cdots\alpha(z)\cdots\bigr)\Bigr) = \\ = \mu\Bigl(\tfrac{\mu}{\alpha}\bigl(\cdots\alpha(x)\cdots\bigr),\tfrac{\mu}{\alpha}\bigl(\cdots\mu(y,z)\cdots\bigr)\Bigr)$$

$\frac{\mu}{\alpha}$ means "$\mu$ or $\alpha$".
The black and blue parts are the same in LHS as in RHS. The blue part consists *only* of the symbols $\alpha$, $\mu$, and parentheses; the number of right and left parentheses there are equal.

**Why do *this*?**

# The structure of hom-associative monomials

Associativity flattens all products: only order of factors matter, bracketing structure does not.

Hom-associativity permits making some rearrangements of bracketing, but we're very far from being able to flatten monomials.
How to map exactly what rearrangement can be done?

One approach for this is to use rewriting to explore the hom-associative operad.

# Operads in universal algebra

A (linear) *operad* $\mathcal{P}$ is a family $\{\mathcal{P}(n)\}_{n=0}^{\infty}$ of linear spaces, closed under composition: each $\mathcal{P}(n)$ may be thought of as a set of multilinear operations with *arity* $n$ (taking $n$ operands).

- Multiplication $\mu$ lives in arity 2 of an operad.
- The hom $\alpha$ lives in arity 1 of an operad.
- Actual algebra elements can be viewed as making up the arity 0 component $\mathcal{P}(0)$.
- Associativity $\mu \circ (\mu \otimes \mathrm{id}) = \mu \circ (\mathrm{id} \otimes \mu)$ and hom-associativity $\mu \circ (\mu \otimes \alpha) = \mu \circ (\alpha \otimes \mu)$ are equalities of arity 3 elements in the operad.

The variety of hom-associative algebras can be formalised as those algebras whose operads of multilinear operations are the codomains of an operad homomorphism from a particular operad,

namely the hom-associative operad $\mathcal{HA}ss$ which is freely generated by two elements $\alpha$ and $\mu$ subject only to the relation $\mu \circ (\mu \otimes \alpha) = \mu \circ (\alpha \otimes \mu)$.

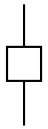
# Graphical notation

Traditional formula notation with nested parentheses, such as

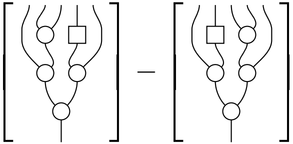
$$(x_1 \cdot (x_2 \cdot x_3)) \cdot (\alpha(x_4) \cdot x_5) - (x_1 \cdot \alpha(x_2)) \cdot ((x_3 \cdot x_4) \cdot x_5),$$

is poorly suited for highlighting patterns in the bracketing *structure* of monomials.

A superior (although more spacious) alternative is to use a graphical notation that shows the expression parse trees directly. Denoting

$\mu$ as  and $\alpha$ as ,

the above hom-algebra expression corresponds to the operad element



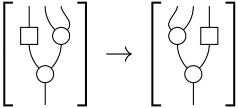
$\in \mathcal{P}(5).$

# Rewriting

Just like systems of polynomial equations correspond to ideals in the polynomial algebra, which may be mapped effectively using Gröbner bases, so may identities such as associativity or hom-associativity be regarded as equations in an operad and their consequences be explored using rewriting.

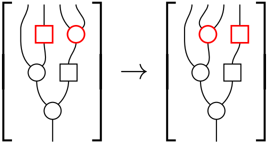
Concretely, one begins with one rule

## Rule 1

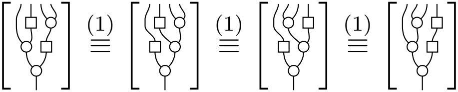


expressing hom-associativity, and runs completion on the rewrite system of that rule. This produces many new rules that are non-obvious logical consequences of hom-associativity.

# Rule 2

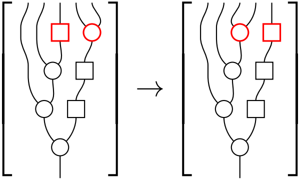


## Proof.

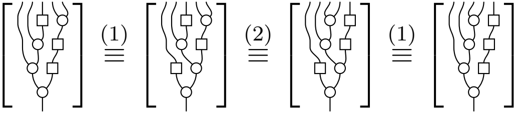


modulo hom-associativity. □

## Rule 3

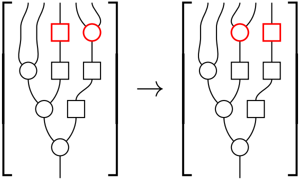


**Proof.**

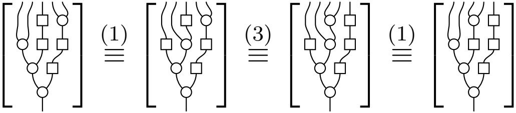



modulo hom-associativity. □

## Rule 4

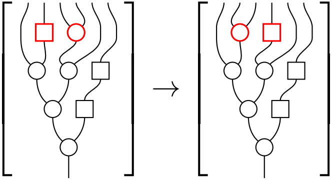


**Proof.**

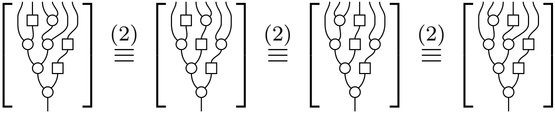



modulo hom-associativity. □

## Rule 5

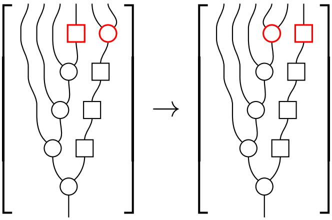


**Proof.**

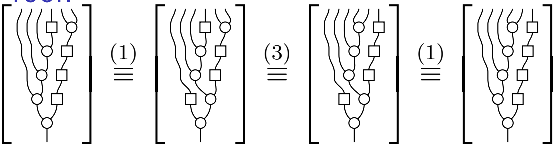



modulo hom-associativity. □

# Rule 6

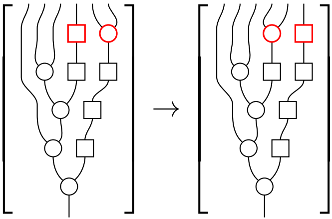


**Proof.**

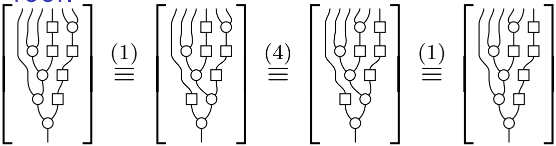



modulo hom-associativity. □

# Rule 7

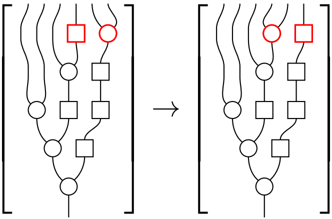


**Proof.**

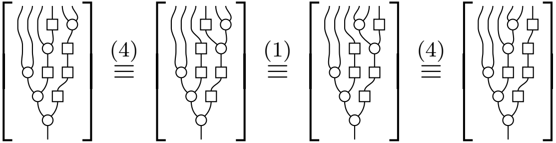



modulo hom-associativity. □

# Rule 8

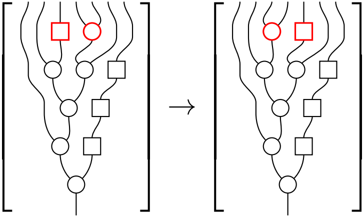


**Proof.**

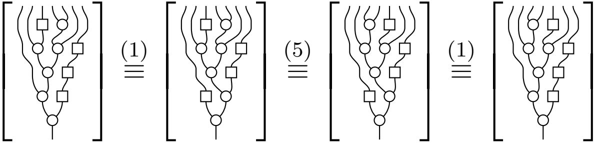



modulo hom-associativity. □

## Rule 9

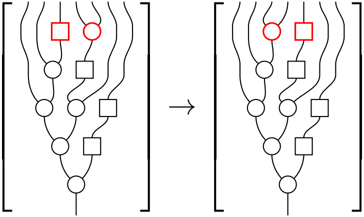


**Proof.**

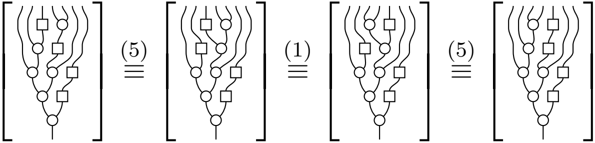



modulo hom-associativity. □

## Rule 10



**Proof.**




modulo hom-associativity. □

# Analysis of calculation results

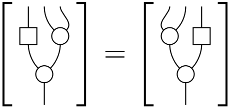
It doesn't stop here; with a few hours' work one can derive several thousand rules, many of which likely belong to infinite rule families.

Quite strikingly, all of these rules only switch places between an $\alpha$ and a $\mu$.
In part this outcome (as in Gröbner basis calculations) depends on the monomial order that was used—another choice of order might produce some other pattern—but the order here is about the simplest one could choose.

The completion has not terminated (it probably *will not*), but we are still able to make conclusions about the complete system thanks to the homogenity of the rules.

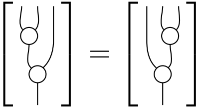
# Homogenity

The hom-associativity identity



is homogeneous in a number of ways:

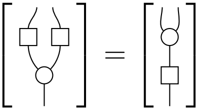
1. Both sides have the same number of $\alpha$s and $\mu$s.
2. Each operand is subjected to the the same number of operations in the left hand side as it is in the right hand side.
3. Every operand in a term is subjected to the same number of operations (namely two).

For contrast, ordinary associativity  is only homogeneous in the first sense. Hom-associativity is what one gets by homogenising it!

# Homogenity of other identities

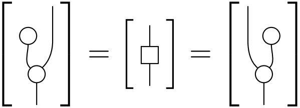
The derived rules shown above are not homogeneous in the third sense, but that is not needed for drawing conclusions from a partial completion procedure.

The multiplicativity (of $\alpha$) identity

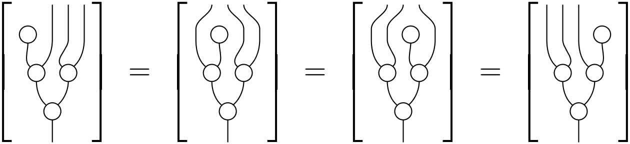



is not homogeneous in the first sense, which complicates (but does not prohibut) drawing conclusions from a partial completion.
More disruptively, it makes the derived rules far more complicated—spoiling the neat pattern from above—so I like to avoid assuming it.

## Hom-unitality



is neatly homogeneous (count the unit  as degree 0). It also offers



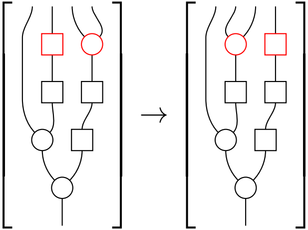
$$(1\cdot x_1)\cdot(x_2\cdot x_3) = (x_1\cdot 1)\cdot(x_2\cdot x_3) = (x_1\cdot x_2)\cdot(1\cdot x_3) = (x_1\cdot x_2)\cdot(x_3\cdot 1)$$

as another axiom which would imply hom-associativity.

# The canyon structure

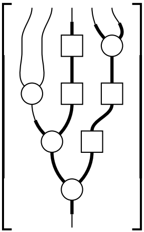
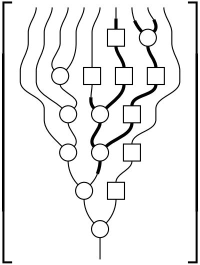
Common for all those rules derived from the hom-associativity axiom is that they all exhibit a canyon structure: the $\alpha$ and $\mu$ being exchanged sit at opposite sides of a "canyon" in the expression, the walls of which may be made up from some arbitrary combination of $\alpha$s and $\mu$s, e.g.:

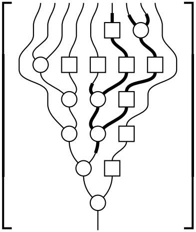
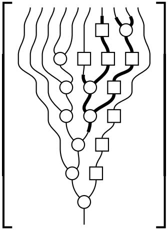
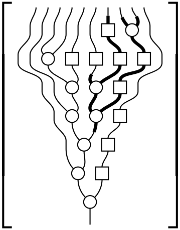
`aa ma` canyon:



This particular rule *does not* follow from hom-associativity, but it appears as the core of a number of rules that do follow …

# LHS of some rules with `aa ma` canyon

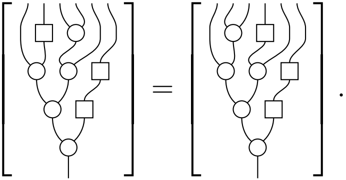
Rule 4:  . Rule 382:  . Rule 383:

 . Rule 386:  . Rule 665:  .

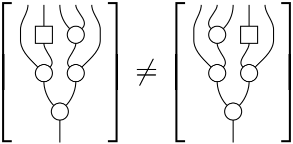
With such "padding" around the canyon, they are logical consequences of hom-associativity. Remove any operation, and they are not!

# The problem with padding

One rule with padding is Rule 5, according to which



The padding is required for its derivation from the hom-associativity axiom. Without that,



This has consequences.

### Example (A free algebra with zero divisors)

Let $(\mathcal{F}, \mu, \alpha)$ be the free hom-associative $\mathbb{Q}$-algebra generated by one element $x$. Then

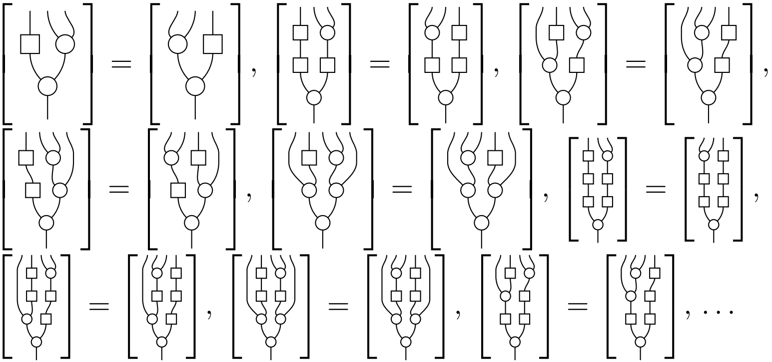
$$d = \mu\Big(\mu\big(x, \alpha(x)\big), \mu\big(\mu(x, x), x\big)\Big) - \mu\Big(\mu\big(x, \mu(x, x)\big), \mu\big(\alpha(x), x\big)\Big)$$

has

$$d \neq 0 \quad \text{but} \quad \mu\big(d, \alpha(\alpha(y))\big) = 0 \quad \text{for all } y \in \mathcal{F}.$$

# Remove the padding!

A hom-algebra is strongly hom-associative if it satisfies the full set of *canyon identities*



An $\alpha$ and a $\mu$ may be switched whenever they are at the opposing edges of a canyon, regardless of the composition of the canyon walls.

# Rewrite aspect

Normal forms of (weakly) hom-associative monomials is difficult to recognise. Canyons can be searched for, but it is hard to tell whether there is a match for the left hand side of a rule, because the padding is difficult to systematise.

The rewrite rules for *strong* hom-associativity are just the canyons. These constitute a confluent rewrite system, so by the Diamond Lemma we get a combinatorial model for the normal form monomials.

The most basic examples are already strongly hom-associative.

### Theorem (Yau twist algebras)

*Let $(\mathcal{A}, \cdot)$ be an associative algebra, let $\alpha\colon \mathcal{A} \longrightarrow \mathcal{A}$ be a homomorphism of that algebra, and let $\mu\colon \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{A}$ be defined by $\mu(x, y) = \alpha(x \cdot y)$ for all $x, y \in \mathcal{A}$. Then $(\mathcal{A}, \mu, \alpha)$ is a strongly hom-associative algebra.*

### Corollary

*If $(\mathcal{A}, \mu, \alpha)$ is a hom-associative algebra such that $\alpha$ is an automorphism of the algebra $(\mathcal{A}, \mu)$, then $(\mathcal{A}, \mu, \alpha)$ is strongly hom-associative.*

The question rather becomes: which hom-associative algebras are *not* strongly hom-associative?

# Open problems

## Non-strongly hom-associative algebras

*Find a concrete example of a hom-associative algebra that is not strongly hom-associative.*

The only example I know so far is that of the free hom-associative algebra, and that's hardly concrete; nobody has yet given a basis for it.

There should exist finite-dimensional examples.

## Canyon egality

*Does every canyon appear as the core of some (weakly) hom-associative identity?*

A 2016 computation enumerating hom-associative identities verified this for all canyons up to height 4, and is only missing two of height 5. (There is much repetition of canyons already seen.)

# Thank you for listening.


- Lars Hellström. Strong hom-associativity.
  Pp. 317–337 in: *Algebraic structures and applications* (SPAS 2017, Västerås and Stockholm, Sweden, October 4–6, Volume 2 of contributions based on the International Conference "Stochastic Processes and Algebraic Structures—From Theory Towards Applications"), editors Sergei Silvestrov, Anatoliy Malyarenko, and Milica Rančić, Springer, Cham, 2020.
  doi: 10.1007/978-3-030-41850-2_12