Torsion free Twisted connections on commutative algebras

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Introduction

Definition

Let $\mathcal{A}$ be an associative algebra and let σ and τ be endomorphisms of $\mathcal{A}$. A $\mathbb{C}$ - linear map $X : \mathcal{A} \to \mathcal{A}$ is called a (σ, τ)-derivation if

$$X(fg) = \sigma(f)X(g) + X(f)\tau(g)$$

for every $f, g \in \mathcal{A}$.

Definition

A (σ, τ)-*algebra* $\Sigma = (\mathcal{A}, \{X_a\}_{a\in I})$ is a pair where $\mathcal{A}$ is an associative algebra (over $\mathbb{C}$) and X_a is a (σ_a, τ_a)-derivation of $\mathcal{A}$ for $a \in I$.

Definition

For a (σ, τ)-algebra $\Sigma = (\mathcal{A}, \{X_a\}_{a\in I})$ we let

$$T\Sigma \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{A})$$

be the vector space generated by $\{X_a\}_{a\in I}$.

Σ-module

Definition

Let $\Sigma = (\mathcal{A}, \{X_a\}_{a \in I})$ be a (σ, τ)-algebra. A *left Σ-module* $(M, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a \in I})$ is a left $\mathcal{A}$-module M together with $\mathbb{C}$-linear maps $\hat{\sigma}_a, \hat{\tau}_a : M \to M$ such that

$$\hat{\sigma}_a(fm) = \sigma_a(f)\hat{\sigma}_a(m)$$
$$\hat{\tau}_a(fm) = \tau_a(f)\hat{\tau}_a(m)$$

for $f \in \mathcal{A}$, $m \in M$ and $a \in I$.

Definition

Let $\Sigma = (\mathcal{A}, \{X_a\}_{a \in I})$ be a (σ, τ)-algebra. A right Σ-module $(M, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a \in I})$ is a right $\mathcal{A}$-module M together with $\mathbb{C}$-linear maps $\hat{\sigma}_a, \hat{\tau}_a : M \to M$ such that

$$\hat{\sigma}_a(mf) = \hat{\sigma}_a(m)\sigma_a(f)$$
$$\hat{\tau}_a(mf) = \hat{\tau}(m)\tau_a(f)$$

for $f \in \mathcal{A}$, $m \in M$ and $a \in I$.

Twisted connection on Σ-modules

Definition

Let $\Sigma = (\mathcal{A}, \{X_a\}_{a\in I})$ be a (σ, τ)-algebra and let $(M, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a\in I})$ be a left Σ-module. A left (σ, τ)-connection on M is a bilinear map $\nabla : T\Sigma \times M \to M$ satisfying

$$\nabla_{X_a}(fm) = \sigma_a(f)\nabla_{X_a} m + X_a(f)\hat{\tau}_a(m).$$

A right (σ, τ)-connection on M is a bilinear map $\nabla : T\Sigma \times M \to M$ satisfying

$$\nabla_{X_a}(mf) = \hat{\sigma}_a(m)X_a(f) + \nabla_{X_a}(m)\tau_a(f).$$

A (σ, τ)-bimodule connection is both left and right (σ, τ)-connection.

Commutative (σ, τ)-algebras

Let $\mathcal{A}$ be a unital commutative algebra and $\sigma, \tau : \mathcal{A} \to \mathcal{A}$ be endomorphisms of $\mathcal{A}$. The linear map $X_f : \mathcal{A} \to \mathcal{A}$ defined by

$$X_f(a) = f\delta(a) = f(\tau(a) - \sigma(a))$$

for all $f \in \mathcal{A}$ and $a \in \mathcal{A}$ is a (σ, τ)-derivation.
Let $\{X_f\}_{f\in\mathcal{A}}$ be a set of (σ, τ)-derivations of $\mathcal{A}$. One has the following:

(a) $\Sigma = (\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ is a (σ, τ)-algebra,

(b) $T\Sigma = \{X_f\}_{f\in\mathcal{A}}$,

(c) $T\Sigma$ is has a $\mathcal{A}$-bimodule structure by setting

$$f \cdot X_g = X_{fg} \quad \text{and} \quad X_g \cdot f = X_{gf}.$$

Free module structure on $T\Sigma$

Definition

The (σ, τ)-algebra $(\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ is regular if there exists $a \in \mathcal{A}$ such that $\delta(a)$ is not a zero divisor.

Proposition

If $\Sigma = (\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ is a regular (σ, τ)-algebra. Then $T\Sigma$ is a free $\mathcal{A}$-bimodule generated by $X_{\mathbb{1}}$.

Proof.

Since every element $X \in T\Sigma$ can be written as $X = X_f$ for some $f \in \mathcal{A}$, one has

$$X = X_f = f \cdot X_{\mathbb{1}},$$

showing that $X_{\mathbb{1}}$ generates $T\Sigma$. Since $(\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ is regular, choose $a_0 \in \mathcal{A}$ such that $\delta(a_0)$ is not a zero divisor, one has

$$f \cdot X_{\mathbb{1}} = 0 \implies f\delta(a_0) = 0 \quad \implies f = 0,$$

Σ-module structure on $T\Sigma$

Proposition

Let $\Sigma = (\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ *be a* (σ, τ)*-algebra. Then* $(T\Sigma, \{(\hat{\sigma}, \hat{\tau})\})$ *is a* Σ*-bimodule where*

$$\hat{\sigma}(X_f) = X_{\sigma(f)} \quad \textit{and} \quad \hat{\tau}(X_f) = X_{\tau(f)}$$

for $X_f \in T\Sigma$.

Proof.

For $f_1, f_2 \in \mathcal{A}$, one has

$$\hat{\sigma}(f_1 X_g f_2) = \hat{\sigma}(X_{f_1 g f_2}) = X_{\sigma(f_1)\sigma(g)\sigma(f_2)} = \sigma(f_1)X_{\sigma(g)}\sigma(f_2)$$
$$= \sigma(f_1)\hat{\sigma}(X_g)\sigma(f_2).$$

Similarly, one can show that $\hat{\tau}(f_1 X_g f_2) = \tau(f_1)\hat{\tau}(X_g)\tau(f_2)$. This shows that $(T\Sigma, \{(\hat{\sigma}, \hat{\tau})\})$ is a Σ-bimodule. $\square$

(σ, τ)-bimodule connection on $T\Sigma$

Proposition

Let $\Sigma = (\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ be a (σ,τ)-algebra. Then there exist a (σ,τ)-bimodule connection $\nabla^0_{X_f} : T\Sigma \to T\Sigma$ on $(T\Sigma, \{(\hat{\sigma}, \hat{\tau})\})$ given by $\nabla^0_{X_f}(X_g) = fX_{\delta(g)}$

Proof.

The left Leibniz rule:

$$\nabla^0_{X_f}(aX_g) = fX_{\delta(ag)} = fX_{\delta(a)\tau(g)+\sigma(a)\delta(g)} = X_f(a)\hat{\tau}(X_g) + \sigma(a)\nabla^0_{X_f}(X_g).$$

The right Leibniz rule:

$$\nabla^0_{X_f}(X_g a) = fX_{\delta(ga)} = fX_{\delta(g)\tau(a)+\sigma(g)\delta(a)} = \nabla^0_{X_f}(X_g)\tau(a) + \hat{\sigma}(X_g)X_f(a).$$

□

(σ, τ)-bimodule connection on $T\Sigma$

Proposition

Let $\Sigma = (\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ be a regular (σ, τ)-algebra, let M be a torsion free module and $(M, \{(\hat{\sigma}, \hat{\tau})\})$ be a Σ-bimodule. Then there exist a unique (σ, τ)-bimodule connection ∇^c given by $\nabla^c_{X_f}(m) = f(\hat{\tau}(m) - \hat{\sigma}(m))$.

Proof.

Let ∇ be a (σ, τ)-bimodule connection on $(M, \{(\hat{\sigma}, \hat{\tau})\})$ and let a be such that $\delta(a)$ is not a zero divisor. One has

$$
\begin{aligned}
0 &= \nabla_{X_f}(am - ma) \\
&= \nabla_{X_f}(am) - \nabla_{X_f}(ma) \\
&= \sigma(a)\nabla_{X_f}(m) + X_f(a)\hat{\tau}(m) - \nabla_{X_f}(m)\tau(a) - \hat{\sigma}(m)X_f(a) \\
&= (\sigma(a) - \tau(a))\nabla_{X_f}(m) + X_f(a)(\hat{\tau}(m) - \hat{\sigma}(m)) \\
&= (\sigma(a) - \tau(a))\nabla_{X_f}(m) + f\delta(a)(\hat{\tau}(m) - \hat{\sigma}(m)) \\
&= (\sigma(a) - \tau(a))\nabla_{X_f}(m) + \delta(a)f(\hat{\tau}(m) - \hat{\sigma}(m)) \\
&= (\sigma(a) - \tau(a))\nabla_{X_f}(m) + (\tau(a) - \sigma(a))f\hat{\delta}(m)
\end{aligned}
$$

Proof.

$$\begin{aligned} &= -(\tau(a)-\sigma(a))\nabla_{X_f}(m) + (\tau(a)-\sigma(a))\nabla^c_{X_f}(m) \\ &= (\tau(a)-\sigma(a))(\nabla^c_{X_f}(m) - \nabla_{X_f}(m)). \end{aligned}$$

Since M is a torsion free module, one has

$$\nabla^c_{X_f}(m) - \nabla_{X_f}(m) = 0,$$

implying that

$$\nabla^c_{X_f}(m) = \nabla_{X_f}(m).$$

□

Corollary

Let $\Sigma = (\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ be a regular (σ,τ)-algebra and ∇ be a left connection on $(T\Sigma, \{(\hat{\sigma},\hat{\tau})\})$. If ∇ is a (σ,τ)-bimodule connection then $\nabla = \nabla^0$.

Torsion free (σ, τ)-connection on $T\Sigma$

Definition

Let $\Sigma = (\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ be a regular and let $(T\Sigma, \{(\hat{\sigma}, \hat{\tau})\})$ be a Σ-module. A (σ, τ)- bracket is a bilinear map $[\cdot,\cdot] : T\Sigma \times T\Sigma \to T\Sigma$ defined by

$$\begin{aligned}[X_f, X_g] &= \hat{\sigma}(X_f) \circ \hat{\tau}(X_g) - \hat{\sigma}(X_g) \circ \hat{\tau}(X_f) \\ &= (\sigma(f)X_{\mathbb{1}}(\tau(g)) - \sigma(g)X_{\mathbb{1}}(\tau(f))) \cdot X_{\mathbb{1}}\end{aligned}$$

for $f, g \in \mathcal{A}$.

Definition

Let $[\cdot,\cdot] : T\Sigma, \times T\Sigma \to T\Sigma$ be a bilinear map called a (σ, τ)-bracket on $T\Sigma$. A (σ, τ)-connection ∇ is torsion free if

$$\nabla_{X_{\sigma(f)}}(X_{\tau(g)}) - \nabla_{X_{\sigma(g)}}(X_{\tau(f)}) - [X_f, X_g] = 0$$

for $X_f, X_g \in T\Sigma$.

(σ, τ)-connection on $T\Sigma$

Proposition

Let $\Sigma = (\mathcal{A}, \{X_f\}_{f\in\mathcal{A}})$ be a (σ, τ)-algebra and let $(T\Sigma, \{(\hat{\sigma}, \hat{\tau})\})$ be a Σ-bimodule with (σ, τ)-braket $[\cdot,\cdot] : T\Sigma, \times T\Sigma \to T\Sigma$ where we choose $\hat{\tau}$ to be the identity map on $T\Sigma$. Then all left (σ, Id)-connections are torsion free.

Proof.

By choosing $\hat{\tau}$ to be the identity map on $T\Sigma$ we have chosen (σ, Id)-derivations on $\mathcal{A}$ and (σ, Id)-connections on $T\Sigma$. Further one bracket on $T\Sigma$ is given by

$$[X_f, X_g] = \hat{\sigma}(X_f) \circ X_g - \hat{\sigma}(X_g) \circ X_f = (\sigma(f)X_{\mathbb{1}}(g) - \sigma(g)X_{\mathbb{1}}(f)) \cdot X_{\mathbb{1}}$$

and torsion free connections satisfy the equation

$$\nabla_{X_{\sigma(f)}}(X_g) - \nabla_{X_{\sigma(g)}}(X_f) - [X_f, X_g] = 0.$$

□

Example

Choose $\mathcal{A} = C(\mathbb{R})$, the algebra of continouos functions on $\mathbb{R}$ and $X_f(g) = f\delta = f(\mathrm{Id} - S)$ where Id is the identity map and S is the shifting operator $S(g)(t) = g(t+1)$ for $g \in C(\mathbb{R})$.
Choose $g(t) = be^{at}$ for $a, b > 0$.

$$\delta(g) = g(t) - g(t+1) = b(1 - e^a)e^{at} < 0$$

for $a > 0$ is not a zero divisor in $C(\mathbb{R})$ because For any continuous function $f(t) \neq 0$

$$b(1 - e^a)e^{at} f(t) \neq 0.$$

Then $\Sigma = (C(\mathbb{R}), \{X_f\}_{f \in C(\mathbb{R})})$ is regular.

outlook

It is interesting to learn that in our case that torsion free connections exist with some little restrictions. It is not suprising that the torsion freeness depends the twist (σ and τ). In this direction we wouldlike to explore We have done more thing among them

- metric (σ, τ)-bimodule connection

And finally we would like to

- generalize the case of torsion and curvature.

Thank you very much for your attention.