Homogeneity in commutative graded rings

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BTH

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Explanation of the title: Graded rings and homogeneity

Definition (Graded ring)

Let G be a group and R a ring. We say that R is a *G-graded ring* if there is a collection of additive subgroups $\{R_g\}_{g \in G}$ satisfying

1. $R = \oplus_{g \in G} R_g$, and
2. $R_g R_h \subseteq R_{gh}$, for all $g, h \in G$.

Definition (Homogeneity)

- An element $r \in R$ is called *homogeneous* if $r \in R_g$ for some $g \in G$.
- An ideal I of R is called *graded* (or *homogeneous*) if $I = \oplus_{g \in G}(I \cap R_g)$.

Acknowledgement and a warning!

This talk is based on joint work with
Abolfazl Tarizadeh (University of Maragheh).
arXiv:2108.10235 [math.AC]

Warning

In this talk **ring** means **commutative ring**!

Outline

1. Classical results for polynomial rings
2. Generalizations of McCoy's theorem
3. Generalizations of Armendariz' theorem
4. The Jacobson radical
5. Idempotents
6. A characterization of totally ordered abelian groups

McCoy's theorem

Theorem (McCoy)

If f is a zero-divisor element of the polynomial ring $R[x]$, then $cf = 0$ for some nonzero $c \in R$.

The following well-known conclusion can also easily be proved directly.

Corollary

If R is an integral domain, then $R[x]$ is an integral domain.

Theorem (McCoy)

If f is a zero-divisor element of the polynomial ring $R[x_1, \ldots, x_d]$, then $cf = 0$ for some nonzero $c \in R$.

Armendariz' theorem

Theorem (Armendariz)

Let f, g be elements of the polynomial ring $R[x]$. Then fg is nilpotent if and only if the product of any coefficient of f with any coefficient of g is nilpotent.

The starting point

Remark

$R[x]$ is naturally $\mathbb{Z}$-graded: $R[X] = \oplus_{n\in\mathbb{Z}} S_n$ where

- $S_n := Rx^n$ for $n \geq 0$, and
- $S_n := \{0\}$ for $n < 0$.

Remark

$S := R[x_1, \ldots, x_d]$ is naturally $\mathbb{Z}^d$-graded:
Indeed, $S = \oplus_{(n_1,\ldots,n_d)\in\mathbb{Z}^d} S_{(n_1,\ldots,n_d)}$ where

- $S_{(n_1,\ldots,n_d)} := Rx_1^{n_1}x_2^{n_2}\ldots x_d^{n_d}$ for $(n_1, \ldots, n_d) \geq 0$,
- $S_{(n_1,\ldots,n_d)} := \{0\}$ whenever $n_i < 0$ for some i.

Question

Can McCoy's and Armendariz' theorems be generalized to the setting of G-graded rings?

An observation

Example

Let G be a group containing an element $g \neq e$ such that $g^n = e$ for some $n \in \mathbb{Z}^+$. Then

$$(g-e)(g^{n-1} + g^{n-2} + \ldots + g^1 + g^0) = 0$$

in any group ring $R[G]$. This means that $g-e$ is a zero-divisor in $R[G]$. But there is no nonzero element $c \in R$ such that $c \cdot (g-e) = 0$.

Conclusion

If we want to generalize McCoy's theorem to G-graded rings, then we need to restrict ourselves to groups which are **torsion-free**!

Totally ordered abelian groups

Definition

An abelian group G is said to be *totally ordered* or *linearly ordered* if it is equipped with a total ordering $\leq$ such that if $a \leq b$ for some $a, b \in G$, then $a + c \leq b + c$ for all $c \in G$.

Example

- $G = (\mathbb{Z}, +)$
- $G = (\mathbb{R}, +)$
- $G = (\mathbb{Z}^d, +)$, with the lexicographical ordering
- $G = (\mathbb{R}^d, +)$, with the lexicographical ordering

Remark

From now on G will denote an arbitrary **totally ordered abelian group**!

Annihilators

Theorem (Tarizadeh & Öinert)

Let I be an ideal of a G-graded ring R. If $\mathrm{Ann}_R(I) \neq 0$, then there exists a nonzero homogeneous $g \in R$ such that $gI = 0$.

Corollary

If f is a zero-divisor element of a G-graded ring R, then there exists a nonzero homogeneous $g \in R$ such that $fg = 0$.

Corollary (McCoy's theorem)

If f is a zero-divisor element of the polynomial ring $R[x_1, \ldots, x_d]$, then $cf = 0$ for some nonzero $c \in R$.

The Jacobson radical and the nilradical

Definition

- The *Jacobson radical of* R:

$$\mathfrak{J}(R) := \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} \mathfrak{m}$$

- The *nilradical of* R:

$$\mathfrak{N}(R) := \bigcap_{\mathfrak{m} \in \mathrm{Prime}(R)} \mathfrak{m}$$

More concretely, $\mathfrak{N}(R) := \{r \in R \mid r^m = 0 \text{ for some } m \in \mathbb{Z}^+\}$.

Remark

$\mathfrak{N}(R)$ is a **graded ideal** of R, and $\mathfrak{N}(R) \subseteq \mathfrak{J}(R)$.

Radical ideals

Definition

An ideal I of a ring R is called a *radical ideal* if $I = \sqrt{I}$, i.e.
$I = \{r \in R \mid r^n \in I \text{ for some } n \in \mathbb{Z}^+\}$.

Remark

$\mathfrak{N}(R)$ and $\mathfrak{J}(R)$ are **radical ideals** of R.

Ideal quotients and graded ideals

Remark

Let I and J be ideals of a ring R. The *ideal quotient*, $I :_R J = \{f \in R : fJ \subseteq I\}$ is an ideal of R containing I.

Remark

If I **and** J are graded ideals of R, then $I :_R J$ is a graded ideal of R.

Theorem (Tarizadeh & Öinert)

Let I be a ***graded radical ideal*** *of a G-graded ring R and J an* ***arbitrary ideal*** *of R. Then $I :_R J$ is a graded ideal.*

Corollary

If I is an ideal of a G-graded ring R, then $\mathfrak{N}(R) :_R I$ is a graded ideal.

Nilpotent elements

Corollary

Let $f = \sum_{i \in G} f_i$ and $g = \sum_{k \in G} g_k$ be elements of a G-graded ring R. Then fg is nilpotent if and only if $f_i g_k$ is nilpotent for all $i, k \in G$.

Corollary (Armendariz' theorem in several variables)

Let R be a ring and let f, g be elements of the polynomial ring $R[x_1, \ldots, x_d]$. Then fg is nilpotent if and only if the product of any coefficient of f with any coefficient of g is nilpotent.

Graded ideals

Theorem (Bergman, Tarizadeh & Öinert)

The Jacobson radical of a G-graded ring is a graded ideal.

Recall that $\mathfrak{J}(R)$ is a radical ideal!

Corollary

If I is an ideal of a G-graded ring R, then $\mathfrak{J}(R) :_R I$ is a graded ideal.

Corollary

Let $f = \sum_{i \in G} f_i$ and $g = \sum_{k \in G} g_k$ be elements of a G-graded ring R. Then $fg \in \mathfrak{J}(R)$ if and only if $f_i g_k \in \mathfrak{J}(R)$ for all $i, k \in G$.

Idempotents

Theorem (Tarizadeh, Öinert)

Every idempotent element of a G-graded ring R is contained in R_0.

Corollary

If R is a G-graded ring such that R_0 has no non-trivial idempotent, then R has no non-trivial idempotent.

Totally ordered abelian groups

Theorem (Tarizadeh & Öinert, Levi)

For an abelian group H the following assertions are equivalent:

1. *H is a totally ordered group.*
2. *H is torsion-free.*
3. *The Jacobson radical of* ***every*** *H-graded ring is a graded ideal.*
4. *Every idempotent of* ***every*** *H-graded ring R is contained in R_0.*

The end

THANK YOU FOR YOUR ATTENTION!