Levi-Civita connections for a class of noncommutative minimal surfaces

Joakim Arnlind
Linköpings universitet

4th workshop of the Swedish Network for Algebra and Geometry
Zoom, 2021-12-03

References

The work I will present today is based on

Levi-Civita connections for a class of noncommutative minimal surfaces
J. A. Int. J. of Geom. Meth. in Mod. Phys., 2021.

together with an application to the results in

Noncommutative Minimal Surfaces
J. A., J. Choe and J. Hoppe. Letters in Mathematical Physics, 2016.

Noncommutative minimal surfaces

The classical theory of minimal surfaces is an old and rich subject, and still quite active.

From a mathematical point of view, it is interesting to investigate if one can develop a parallel theory in noncommutative geometry.

Analogues of minimal surface equations appear as equations of motion in physical models; e.g. in Membrane and String theory one finds that the (operators corresponding to the) embedding coordinates have to be "harmonic".

Minimal surfaces is a metric theory; i.e. it is concerned with stationary points of the total area integral of an embedded manifold.

Can one find a nice metric theory for noncommutative minimal surfaces? In particular, can one find Levi-Civita connections?

Let us start by studying connections on modules equipped with a hermitian form.

Hermitian forms

In the following, $\mathcal{A}$ is a (noncommutative) unital $*$-algebra and all modules are assumed to be right $\mathcal{A}$-modules.

For noncommutative $*$-algebras, it is natural to replace the metric with its corresponding hermitian form:

Definition

Let $\mathcal{A}$ be a $*$-algebra and let M be a right A-module. A *hermitian form on* M is a map $h : M \times M \to \mathcal{A}$ satisfying

1. $h(m_1, m_2)^* = h(m_2, m_1)$
2. $h(m_1, m_2 a) = h(m_1, m_2)a$
3. $h(m_1 + m_2, m) = h(m_1, m) + h(m_2, m)$

for all $m, m_1, m_2 \in M$ and $a \in \mathcal{A}$.

Invertible Hermitian forms

To a hermitian form h on a right $\mathcal{A}$-module M, one associates the map $\hat{h} : M \to M^*$ given by $\hat{h}(m_1)(m_2) = h(m_1, m_2)$.

Definition

A hermitian form h on the $\mathcal{A}$-module M is called *invertible* if $\hat{h} : M \to M^*$ is a bijection, and one defines $h^{-1} : M^* \times M^* \to \mathcal{A}$ as

$$h^{-1}(\omega_1, \omega_2) = \omega_1\big(\hat{h}^{-1}(\omega_2)\big).$$

For instance, on a free right $\mathcal{A}$-module $\mathcal{A}^n$ with basis $\{\hat{e}_i\}_{i=1}^n$, a hermitian form is determined by $h_{ij} \in \mathcal{A}$ (for $i, j = 1, \ldots, n$) as

$$h(\hat{e}_i U^i, \hat{e}_i V^i) = (U^i)^* h_{ij} V^j.$$

In this case, the hermitian form h is invertible if there exists h^{ij} such that $h^{ij} h_{jk} = \delta^i_k \mathbb{1}$.

Hermitian modules

Definition

Let M be a $\mathcal{A}$-module and let h be a hermitian form on M. The pair (M, h) is called a *hermitian module*.

Definition

A homomorphism of hermitian modules $\phi : (M, h) \to (M', h')$ is a module homomorphism $\phi : M \to M'$ such that $h(m_1, m_2) = h'(\phi(m_1), \phi(m_2))$ for all $m_1, m_2 \in M$.

If $\phi : (M, h) \to (M', h')$ is a module isomorphism then ϕ is called an *isometry* and (M, h) and (M', h') are said to be *isometric* and we write $(M, h) \simeq (M', h')$.

Regular hermitian modules

In the following we shall study connections and curvature on hermitian modules. However, a few regularity assumptions are needed; we assume the metric to be invertible and the module to be a finitely generated projective module.

Definition

If h is an invertible hermitian form on a finitely generated projective $\mathcal{A}$-module M, then (M, h) is called a *regular hermitian $\mathcal{A}$-module*.

Characterization of regular hermitian modules

Given a hermitian module (M, h) and a set of generators $\{e_a\}_{a=1}^n$ one defines $h_{ab} = h(e_a, e_b)$. However, unless the module is free, one cannot find h^{ab} such that $h^{ab}h_{bc} = \delta^a_c\mathbb{1}$.

For regular hermitian modules it turns out that one can find an inverse relative to the generators; i.e. h^{ab} such that $e_a h^{ab} h_{bc} = e_c$. Interestingly, this also characterizes regular hermitian modules.

Proposition

Let (M, h) be a hermitian $\mathcal{A}$-module with generators $\{e_a\}_{a=1}^n$, and set $h_{ab} = h(e_a, e_b)$. Then (M, h) is a regular hermitian module if and only if there exist $h^{ab} \in \mathcal{A}$ (for $a, b = 1, \ldots, n$) such that $(h^{ab})^* = h^{ba}$ and $e_r h^{rs} h_{sc} = e_c$ for $a, b, c = 1, \ldots, n$.

Construction of regular hermitian modules

For a free module $\mathcal{A}^n$, with basis $\{\hat{e}_i\}_{i=1}^n$, one can construct a regular hermitian module by choosing an invertible matrix h_{ij} (with entries in $\mathcal{A}$) and setting

$$h(\hat{e}_i U^i, \hat{e}_j V^j) = (U^i)^* h_{ij} V^j.$$

Then $(\mathcal{A}^n, h)$ is a regular hermitian module. By using orthogonal projections, one can easily construct regular hermitian projective modules.

Proposition

Let $(\mathcal{A}^n, \tilde{h})$ be a free regular hermitian module. If p is an orthogonal projection on $(\mathcal{A}^n, \tilde{h})$, then $(p(\mathcal{A}^n), \tilde{h}|_{p(\mathcal{A}^n)})$ is a regular hermitian module.

Question: Can every regular hermitian module be constructed in this way?

Theorem

Let (M, h') be a regular hermitian right $\mathcal{A}$-module such that M is generated by n elements. Define the following invertible hermitian form[a]

$$b : \mathcal{A}^n \oplus (\mathcal{A}^n)^* \times \mathcal{A}^n \oplus (\mathcal{A}^n)^* \to \mathcal{A}$$
$$b\big((U, \omega), (V, \eta)\big) = \eta(U)^* + \omega(V)$$

for $U, V \in \mathcal{A}^n$ and $\omega, \eta \in (\mathcal{A}^n)^*$. Then there exists a projection

$$\hat{p} : \mathcal{A}^n \oplus (\mathcal{A}^n)^* \to \mathcal{A}^n \oplus (\mathcal{A}^n)^*,$$

which is orthogonal with respect to b, such that

$$\big(\hat{p}(\mathcal{A}^n \oplus (\mathcal{A}^n)^*), b|_{\hat{p}(\mathcal{A}^n \oplus (\mathcal{A}^n)^*)}\big) \simeq (M, h').$$

[a] Note that in the direct sum $\mathcal{A}^n \oplus (\mathcal{A}^n)^*$, the dual $(\mathcal{A}^n)^*$ is considered as a right $\mathcal{A}$-module via $\omega \cdot a = a^* \omega$.

Lie pairs and connections

Let us now consider connections on hermitian modules, which will be done in the context of derivation based differential calculus. In order to simplify the notation, let us introduce:

Definition

A *Lie pair* is a pair $(\mathcal{A}, \mathfrak{g})$ where $\mathcal{A}$ is a unital $*$-algebra and $\mathfrak{g}$ is a $*$-closed Lie algebra of derivations on $\mathcal{A}$.

Definition

Let $(\mathcal{A}, \mathfrak{g})$ be a Lie pair and let M be a right $\mathcal{A}$-module. A *connection on* M is a map $\nabla : \mathfrak{g} \times M \to M$ such that

1. $\nabla_\partial(m_1 + m_2) = \nabla_\partial m_1 + \nabla_\partial m_2$
2. $\nabla_{\lambda_1\partial_1+\lambda_2\partial_2} m = \lambda_1 \nabla_{\partial_1} m + \lambda_2 \nabla_{\partial_2} m$
3. $\nabla_\partial(ma) = (\nabla_\partial m)a + m\partial(a)$

for $m, m_1, m_2 \in M$, $\partial, \partial_1, \partial_2 \in \mathfrak{g}$, $\lambda_1, \lambda_2 \in \mathbb{C}$ and $a \in \mathcal{A}$.

Hermitian connections

Definition

Let $(\mathcal{A}, \mathfrak{g})$ be a Lie pair and let (M, h) be a right hermitian $\mathcal{A}$-module. A connection ∇ on M is called *hermitian* if

$$\partial h(m_1, m_2) = h(\nabla_{\partial^*} m_1, m_2) + h(m_1, \nabla_{\partial} m_2)$$

for all $m_1, m_2 \in M$ and $\partial \in \mathfrak{g}$. A hermitian connection ∇ on a hermitian module (M, h) is said to be *compatible with h*.

Proposition

Let $(\mathcal{A}, \mathfrak{g})$ be a Lie pair and let ∇ be a hermitian connection on the free hermitian module $(\mathcal{A}^n, h)$. If $p : \mathcal{A}^n \to \mathcal{A}^n$ is an orthogonal projection, then $p \circ \nabla$ is a hermitian connection on $(p(\mathcal{A}^n), h|_{p(\mathcal{A}^n)})$.

Hermitian connections on free modules

One can easily derive an explicit expression for hermitian connections on a free module.

Proposition

Let $(\mathcal{A}, \mathfrak{g})$ be a Lie pair, let $\{\partial_a\}_{a=1}^m$ be a hermitian basis of $\mathfrak{g}$ and let $\{\hat{e}_i\}_{i=1}^n$ be a basis of the free module $\mathcal{A}^n$. A connection ∇ on the regular right hermitian $\mathcal{A}$-module $(\mathcal{A}^n, h)$ is hermitian if and only if there exist $\gamma_{a,ij} \in \mathcal{A}$ (for $a = 1, \ldots, m$ and $i, j = 1, \ldots, n$) such that $\gamma_{a,ij}^* = \gamma_{a,ji}$ and

$$\nabla_{\partial_a} \hat{e}_i = \hat{e}_j h^{jk} \left(\tfrac{1}{2} \partial_a h_{ki} + i \gamma_{a,ki} \right) \tag{1}$$

where $h_{ij} = h(\hat{e}_i, \hat{e}_j)$ and $h^{ij} h_{jk} = \delta^i_k \mathbb{1}$.

Combining this with the previous results, one obtains:

Corollary

Let $(\mathcal{A}, \mathfrak{g})$ be a Lie pair and let (M, h) be a regular hermitian $\mathcal{A}$-module. Then there exists a hermitian connection on (M, h).

"Embedded noncommutative manifolds"

Definition

A triple $\Sigma = (\mathcal{A}, \mathfrak{g}, \{X^1, \ldots, X^n\})$, where $(\mathcal{A}, \mathfrak{g})$ is a Lie pair and $X^1, \ldots, X^n \in \mathcal{A}$ are hermitian elements, is called an *embedded noncommutative manifold*.

Given an embedded noncommutative manifold $\Sigma = (\mathcal{A}, \mathfrak{g}, \{X^1, \ldots, X^n\})$ and a basis $\{\hat{e}_i\}_{i=1}^n$ of the free module $\mathcal{A}^n$, we define $\varphi : \mathfrak{g} \to \mathcal{A}^n$ by

$$\varphi(\partial) = \partial X = \hat{e}_i \partial X^i \qquad (2)$$

for $\partial \in \mathfrak{g}$. The (right) module $T\Sigma$ generated by the image of φ will be referred to as the *module of vector fields of* Σ. Recall that since $\mathfrak{g}$ is $*$-closed there exists a basis of $\mathfrak{g}$ consisting of hermitian derivations $\{\partial_a\}_{a=1}^m$. The module $T\Sigma$ is clearly generated by $\{\varphi(\partial_a)\}_{a=1}^m$, and we shall in the following write $e_a = \varphi(\partial_a) = \hat{e}_i \partial_a X^i$, as well as $\partial_a e_b = \hat{e}_i \partial_a \partial_b X^i$.

Torsion free connections

Furthermore, let $h_0 : \mathcal{A}^n \times A^n \to \mathcal{A}$ denote the hermitian form given by

$$h_0(U,V) = \sum_{i=1}^{m} (U^i)^* V^i$$

for $U = \hat{e}_i U^i$ and $V = \hat{e}_i V^i$, and the restriction of h_0 to $T\Sigma$ will be denoted by h. We shall in the following assume that $(T\Sigma, h)$ is a regular hermitian module.

Definition

A connection ∇ on $T\Sigma$ is called *torsion free* if

$$\nabla_{\partial_1}\varphi(\partial_2) - \nabla_{\partial_2}\varphi(\partial_1) = \varphi([\partial_1, \partial_2])$$

for all $\partial_1, \partial_2 \in \mathfrak{g}$. Furthermore, a torsion free hermitian connection on $(T\Sigma, h)$ is called a *Levi-Civita connection*.

The existence of a torsion free and hermitian connection

Theorem

Let $\Sigma = (\mathcal{A}, \mathfrak{g}, \{X^1, \dots, X^n\})$ be a regular embedded noncommutative manifold and let $\{\partial_a\}_{a=1}^m$ be a hermitian basis of $\mathfrak{g}$. For any $\gamma_{a,ij} \in \mathcal{A}$ such that $\gamma_{a,ij}^* = \gamma_{a,ji}$ and $\gamma_{a,bc} = \gamma_{c,ba}$, where $\gamma_{a,bc} = (e_b^i)^* \gamma_{a,ij} e_c^j$,

$$\nabla_{\partial_a}(e_b m^b) = e_b \partial_a m^b + e_c h^{cp}\big(h^0(e_p, \partial_a e_b) + i\gamma_{a,pb}\big) m^b \tag{3}$$

defines a Levi-Civita connection on $(T\Sigma, h)$. In particular, choosing $\gamma_{a,ij} = 0$, we conclude that there exists a Levi-Civita connection on every regular embedded noncommutative manifold.

Noncommutative minimal surfaces

The *Weyl algebra* is the unital algebra generated by elements U, V satisfying $[U, V] = i\hbar\mathbb{1}$. The Weyl algebra satisfies the *Ore condition*, which implies that it can be embedded in a field of fractions $\mathfrak{F}_\hbar$. The Weyl algebra (and its field of fractions) can be equipped with a $*$-algebra structure by letting $U^* = U$ and $V^* = V$.

Let $\mathfrak{g}_2$ denote the (abelian) Lie algebra generated by the two hermitian (inner) derivations $\hat{\partial}_u(A) = [A, V]/(i\hbar), \hat{\partial}_v(A) = [A, U]/(i\hbar)$.

Definition

An embedded noncommutative manifold $\Sigma = (\mathfrak{F}_\hbar, \mathfrak{g}_2, \{X^1, \ldots, X^n\})$ is called a *noncommutative minimal surface* if

$$\hat{\partial}_u^2 X^i + \hat{\partial}_v^2 X^i = -\frac{1}{\hbar^2}\big[[X^i, V], V\big] - \frac{1}{\hbar^2}\big[[X^i, U], U\big] = 0 \quad (i = 1, \ldots, n)$$

$$h(e_u, e_u) = h(e_v, e_v) \equiv \mathcal{E} \qquad - h(e_u, e_v)^* = h(e_u, e_v) = i\mathcal{F}$$

$e_u = \hat{\partial}_u X$, $e_v = \hat{\partial}_v X$, $h(Y, Z) = \sum_{i=1}^n (Y^i)^* Z^i$.

Thus, for a noncommutative minimal surface, the metric is given by

$$h_{ab} = h(e_a, e_b) = \begin{pmatrix} \mathcal{E} & i\mathcal{F} \\ -i\mathcal{F} & \mathcal{E} \end{pmatrix}$$

with $\mathcal{E}^* = \mathcal{E}$ and $\mathcal{F}^* = \mathcal{F}$. Since $\mathfrak{F}_\hbar$ is a field, as long as $\mathcal{E} + \mathcal{F} \neq 0$ and $\mathcal{E} - \mathcal{F} \neq 0$, it is clear that the metric is invertible, with inverse

$$h^{-1} = (\mathcal{E} + \mathcal{F})^{-1} \begin{pmatrix} \mathcal{E} & -i\mathcal{F} \\ i\mathcal{F} & \mathcal{E} \end{pmatrix} (\mathcal{E} - \mathcal{F})^{-1},$$

implying that $T\Sigma$ is a free module and that $(\mathfrak{F}_\hbar, \mathfrak{g}_2, \{X^1, \ldots, X^n\})$ is a regular embedded noncommutative manifold. Writing

$$\Phi_1 = \Phi = \partial X = \tfrac{1}{2}(e_u - ie_v) \qquad \Phi_2 = \bar{\Phi} = \bar{\partial} X = \tfrac{1}{2}(e_u + ie_v).$$

the metric becomes

$$H_{ab} = h(\Phi_a, \Phi_b) = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathcal{E} + \mathcal{F} & 0 \\ 0 & \mathcal{E} - \mathcal{F} \end{pmatrix}.$$

Levi-Civita connections

For arbitrary $\tilde{\gamma}_1, \tilde{\gamma}_2 \in \mathfrak{F}_\hbar$ the following defines a hermitian and torsion free connection on the module $T\Sigma$ generated by $\Phi, \bar{\Phi}$:

$$\begin{aligned}
\nabla_\partial \Phi &= \Phi S^{-1}\partial S + i\Phi S^{-1}\tilde{\gamma}_1 + i\bar{\Phi} T^{-1}\tilde{\gamma}_2 \\
\nabla_{\bar{\partial}} \bar{\Phi} &= \bar{\Phi} T^{-1}\bar{\partial} T + i\Phi S^{-1}\tilde{\gamma}_2^* + i\bar{\Phi} T^{-1}\tilde{\gamma}_1^* \\
\nabla_\partial \bar{\Phi} &= i\Phi S^{-1}\tilde{\gamma}_1^* + i\bar{\Phi} T^{-1}\tilde{\gamma}_1 = \nabla_{\bar{\partial}}\Phi.
\end{aligned}$$

For instance, choosing $\tilde{\gamma}_1 = \tilde{\gamma}_2 = 0$, one obtains

$$\nabla_\partial \Phi = \Phi S^{-1}\partial S \qquad \nabla_{\bar{\partial}} \bar{\Phi} = \bar{\Phi} T^{-1}\bar{\partial} T \qquad \nabla_\partial \bar{\Phi} = \nabla_{\bar{\partial}}\Phi = 0$$

giving the curvature

$$\begin{aligned}
R(\partial, \bar{\partial})\Phi &= \nabla_\partial\nabla_{\bar{\partial}}\Phi - \nabla_{\bar{\partial}}\nabla_\partial\Phi = -\Phi\bar{\partial}\big(S^{-1}\partial S\big) \\
R(\partial, \bar{\partial})\bar{\Phi} &= \nabla_\partial\nabla_{\bar{\partial}}\bar{\Phi} - \nabla_{\bar{\partial}}\nabla_\partial\bar{\Phi} = \bar{\Phi}\partial\big(T^{-1}\bar{\partial} T\big).
\end{aligned}$$

Thank you!

Algebraic minimal surfaces

For instance, for arbitrary polynomial $F(\Lambda)$ the following defines a minimal surface in $\mathcal{A}_\hbar^3$:

$$X^1 = \mathrm{Re}\left((\mathbb{1} - \Lambda^2)\partial^2 F(\Lambda) + 2\Lambda\partial F(\Lambda) - 2F(\Lambda)\right)$$
$$X^2 = \mathrm{Re}\left(i(\mathbb{1} + \Lambda^2)\partial^2 F(\Lambda) - 2i\Lambda\partial F(\Lambda) + 2iF(\Lambda)\right)$$
$$X^3 = \mathrm{Re}\left(2\Lambda\partial^2 F(\Lambda) - 2\partial F(\Lambda)\right)$$

with $\Lambda = U + iV$ and $\partial(A) = [A, \Lambda^*]/(2\hbar)$. In other words, the above elements satisfy (for $i = 1, 2, 3$)

$$\big[[X^i, U], U\big] + \big[[X^i, V], V\big] = 0.$$

The simplest case is the noncommutative Enneper surface:

$$X^1 = U + UV^2 - \frac{1}{3}U^3 - i\hbar V$$
$$X^2 = -V - U^2 V + \frac{1}{3}V^3 + i\hbar U$$
$$X^3 = U^2 - V^2.$$