Multiplication and linear integral operators on L_p spaces representing polynomial covariant type commutation relations

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Introduction

In many areas of applications there may be found relations of the form

$$AB = BF(A) \tag{1}$$

for a certain function F satisfying certain conditions where A, B are elements of an associative algebra over a field (for example, field of complex numbers).
This relation appears in Quantum Mechanics, Wavelet Analysis, and have some connection with Dynamical Systems and for specific spaces it is related to Spectral Theory.

Introduction cont.

A pair (A, B) of the corresponding associative algebra that satisfies (1) called a representation of this relation. One of the main objectives is to find representations of relation and study their properties. We construct representations of Relation (1) by linear integral and multiplication operators on L_p spaces.

Proposition

Let $A: L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B: L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, be defined as follows

$$(Ax)(t) = \int_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

almost everywhere, where $k(t,s): \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, is a measurable function, satisfying

$$\int_{\mathbb{R}} \left(\int_{\alpha}^{\beta} |k(t,s)|^q ds \right)^{p/q} dt < \infty, \tag{2}$$

$1 < q < \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$ *and* $b \in L_\infty(\mathbb{R})$.

Consider a real valued polynomial $F(t) = \delta_0 + \delta_1 t + \delta_2 t^2 + \ldots + \delta_n t^n$, where $\delta_0, \delta_1, \ldots, \delta_n$ are real constants. We set

$$k_0(t,s) = k(t,s),\ k_m(t,s) = \int\limits_{\alpha}^{\beta} k(t,\tau) k_{m-1}(\tau,s) d\tau, \quad m = \overline{1,n}$$

$$F_n(k(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t,s), \quad n \in \mathbb{N}. \tag{3}$$

Then, $AB = BF(A)$ if and only if for all $x \in L_p(\mathbb{R})$

$$b(t)\delta_0 x(t) + b(t)\int_{\alpha}^{\beta} F_n(k(t,s))x(s)ds = \int_{\alpha}^{\beta} k(t,s)b(s)x(s)ds. \tag{4}$$

If $\delta_0 = 0$, that is, $F(t) = \delta_1 t + \delta_2 t^2 + \ldots + \delta_n t^n$ then the condition (4) reduces to the following: for almost every (t,s) in $\mathbb{R} \times [\alpha, \beta]$

$$b(t)F_n(k(t,s)) = k(t,s)b(s). \tag{5}$$

Corollary

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, *be defined as follows*

$$(Ax)(t) = \int\limits_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

almost everywhere, where $k(t,s) : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, *is a measurable function satisfying* (2), $b \in L_\infty(\mathbb{R})$ *nonzero such that the set*

$$\mathrm{supp}\, b \cap [\alpha, \beta]$$

has measure zero. Consider a real valued polynomial $F(t) = \delta_0 + \delta_1 t + \delta_2 t^2 + \ldots + \delta_n t^n$, *where* $\delta_0, \ldots, \delta_n$ *are real constants.*

We set

$$k_0(t,s)=k(t,s),\ k_m(t,s) = \int\limits_{\alpha}^{\beta} k(t,\tau)k_{m-1}(\tau,s)d\tau, \quad m=\overline{1,n}$$

$$F_n(k(t,s)) = \sum_{j=1}^{n} \delta_j k_{j-1}(t,s), \quad n\in\mathbb{N}.$$

Then, we have $AB=BF(A)$ if and only if $\delta_0=0$ and the set

$$(\mathrm{supp}\, b\times[\alpha,\beta])\cap \mathrm{supp}\, g_{Fk}$$

has measure zero in $\mathbb{R}\times[\alpha,\beta]$, where $g_{Fk}:\mathbb{R}\times[\alpha,\beta]\to\mathbb{R}$ defined by $g_{Fk}(t,s)=F_n(k(t,s))$.

Corollary

Let $A, B : L_p([-M, M]) \to L_p([-M, M])$ *be nonzero operators defined as follows*

$$(Ax)(t) = \int\limits_{\alpha}^{\beta} k(t,s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

almost everywhere, where $\alpha, \beta \in \mathbb{R}$, $M = \max\{|\alpha|, |\beta|\}$, $k(t,s) : [-M, M] \times [\alpha, \beta] \to \mathbb{R}$, $b : [-M, M] \to \mathbb{R}$ *are given by*

$$k(t,s) = a_0 + a_1 t + c_1 s, \qquad b(t) = b_0 + b_1 t + b_2 t^2,$$

a_0, a_1, b_0, b_1, b_2, c_1 *are real numbers. Consider a polynomial* $F : \mathbb{R} \to \mathbb{R}$ *defined by* $F(t) = \delta_1 t + \delta_2 t^2$, *where* δ_1, δ_2 *are real numbers.*

Then, we have $AB = BF(A)$ if and only if for almost every (t, s) in $\mathbb{R} \times [\alpha, \beta]$

$$b(t)F_2(k(t,s)) = k(t,s)b(s)$$

which it is equivalent to $b(s) \equiv b_0$ non-zero constant. In particular, one of the following cases holds:

1 If $a_1 = c_1 = 0$ and $\delta_2 \neq 0$, then

$$a_0 = \frac{1-\delta_1}{\delta_2(\beta-\alpha)}.$$

Otherwise if $\delta_2 = 0$ then $\delta_1 = 1$ and a_0 is free.

2 if $a_1 = 0$ and $\delta_2 \neq 0$ then

$$a_0 = \frac{2 - 2\delta_1 - \delta_2 c_1(\beta^2 - \alpha^2)}{2\delta_2(\beta - \alpha)},$$

c_1 is free. If $a_0 = 0$, $\beta \neq -\alpha$ then

$$c_1 = \frac{2 - 2\delta_1}{\delta_2(\beta^2 - \alpha^2)}.$$

Otherwise if $\delta_2 = 0$ or $\beta = -\alpha$ then a_0, c_1 are free and $\delta_1 = 1$.

3 $c_1 = 0$ and $\delta_2 \neq 0$ then

$$a_0 = \frac{2 - 2\delta_1 - \delta_2 a_1(\beta^2 - \alpha^2)}{2\delta_2(\beta - \alpha)},$$

a_1 is free. If $a_0 = 0$, $\beta \neq -\alpha$ then

$$a_1 = \frac{2 - 2\delta_1}{\delta_2(\beta^2 - \alpha^2)}.$$

Otherwise if $\delta_2 = 0$ or $\beta = -\alpha$ then a_0, a_1 are free and $\delta_1 = 1$.

Corollary

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, be defined as follows

$$(Ax)(t) = \int_{\alpha}^{\beta} a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

almost everywhere, where $a \in L_p(\mathbb{R})$, $c \in L_q([\alpha, \beta])$ $(\alpha, \beta \in \mathbb{R})$, $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ and $b \in L_\infty(\mathbb{R})$. Consider a polynomial $F : \mathbb{R} \to \mathbb{R}$ defined by $F(t) = \delta_1 t + \delta_2 t^2 + \ldots + \delta_n t^n$, where $\delta_1, \ldots, \delta_n$ are real constants. We set

$$\mu = \int_{\alpha}^{\beta} a(s)c(s)ds.$$

Then, we have $AB = BF(A)$ if and only if the set

$$\operatorname{supp} g_{ac} \cap \operatorname{supp} g_b,$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$, where g_{ac}, $g_b : \mathbb{R} \times [\alpha, \beta] \to \mathbb{R}$ are defined as follows

$$\begin{aligned} g_{ac}(t,s) &= a(t)c(s) \\ g_b(t,s) &= b(t)\sum_{j=1}^{n} \delta_j \mu^{j-1} - b(s). \end{aligned}$$

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows

$$(Ax)(t) = \int_0^2 a(t)c(s)x(s)ds, \quad (Bx)(t) = b(t)x(t),$$

almost everywhere, where $a(t) = I_{[0,1]}(t)(1+t^2)$, $c(s) = 1$, $b(t) = I_{[1,2]}(t)t^2$. Consider a polynomial $F : \mathbb{R} \to \mathbb{R}$ defined by $F(t) = \delta_1 t + \delta_2 t^2 + \ldots + \delta_n t^n$, where $\delta_1, \ldots, \delta_n$ are real constants. Then, operators A and B satisfy the relation

$$AB = BF(A).$$

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$, $\alpha, \beta \in \mathbb{R}$ be defined as follows

$$(Ax)(t) = \int_{\alpha}^{\beta} I_{[\alpha,\beta]}(t)x(s)ds, \quad (Bx)(t) = I_{[\alpha,\beta]}(t)x(t),$$

almost everywhere. Let $F : \mathbb{R} \to \mathbb{R}$, $F(t) = \delta_1 t + \delta_2 t^2 + \ldots + \delta_n t^n$, where $\delta_1, \ldots, \delta_n$ are constants. Then, operators A and B satisfy

$$AB = BF(A)$$

if and only if

$$\sum_{j=1}^{n} \delta_j (\beta - \alpha)^{j-1} = 1.$$

Proposition

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ *be defined as follows*

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} k(t,s)x(s)ds$$

almost everywhere, where $a \in L_\infty(\mathbb{R})$, $k(t,s) : \mathbb{R} \times [\alpha,\beta] \to \mathbb{R}$, $\alpha, \beta \in \mathbb{R}$, *is a Lebesgue measurable function satisfying* (2). *For a polynomial* $F : \mathbb{R} \to \mathbb{R}$ *defined by* $F(t) = \delta_0 + \delta_1 t + \delta_2 t^2 + \ldots + \delta_n t^n$, *where* $\delta_0, \delta_1, \ldots, \delta_n$ *are constants.*

Then

$$AB = BF(A)$$

if and only if the set

$$\operatorname{supp} g_{aF} \cap \operatorname{supp} k$$

has measure zero in $\mathbb{R} \times [\alpha, \beta]$, where $g_{aF}(t,s) = a(t) - F(a(s))$.

Example

Let $A : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $B : L_p(\mathbb{R}) \to L_p(\mathbb{R})$, $1 < p < \infty$ be defined as follows

$$(Ax)(t) = a(t)x(t), \quad (Bx)(t) = \int_{\alpha}^{\beta} b(t)c(s)x(s)ds$$

almost everywhere, where $a(t) = \gamma_0 + I_{[\alpha, \frac{\alpha+\beta}{2}]}(t)t^2$, γ_0 is a real number, $b(t) = (1 + t^2)I_{[\beta+1,\beta+2]}(t)$, $c(s) = I_{[\frac{\alpha+\beta}{2},\beta]}(s)(1 + s^4)$, $\alpha, \beta \in \mathbb{R}$. Let $F : \mathbb{R} \to \mathbb{R}$, $F(t) = \delta_0 + \delta_1 t$, where $\delta_0, \delta_1 \in \mathbb{R}$ and $\delta_1 \neq 0$. If $\delta_0 = \gamma_0 - \delta_1\gamma_0$ then the above operators satisfy the relation

$$AB - \delta_0 BA = \delta_1 B.$$

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Thank you!!!