Prime Leavitt path algebras via nearly epsilon-strongly graded rings

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Reference

D. Lännström, P. Lundström, J. Öinert, S. Wagner. *Prime group graded rings with applications to partial crossed products and Leavitt path algebras*. Preprint.

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Group graded rings

Definition

Let G be a group and let S be a ring. A *grading* of S is a collection of additive subsets of S, $\{S_g\}_{g \in G}$, such that

$$S = \bigoplus_{g \in G} S_g,$$

and $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. The ring S is called a *G-graded ring*. S_e is called the *principal component*.

Graded rings: Examples I

Example

The Laurent polynomial ring is $\mathbb{Z}$-graded by,

$$R[x, x^{-1}] = \bigoplus_{i \in \mathbb{Z}} Rx^i.$$

Example

(The group ring) Let G be a group and let R be a unital ring. The group ring $R[G] = \bigoplus_{g \in G} R\delta_g$ is naturally G-graded.

Strongly graded rings

Definition

A G-grading $\{S_g\}$ of a ring S is called *strong* if $S_g S_h = S_{gh}$ holds for all $g, h \in G$. The ring S is called *strongly* G-graded.

Example

Let R be a unital ring. The Laurent polynomial ring $R[x, x^{-1}] = \bigoplus_{i \in \mathbb{Z}} Rx^i$ is strongly $\mathbb{Z}$-graded.

Example

Let G be a group and let R be a unital ring. The group ring $R[G]$ is strongly G-graded.

Nearly epsilon-strongly graded rings

Definition (Nystedt, Öinert and Pinedo 2016)

Let S be a G-graded ring. If, for every $g \in G$,

1. $S_g = S_g S_{g^{-1}} S_g$, and,
2. $S_g S_{g^{-1}}$ is a unital ideal of S_e,

then S is called *epsilon-strongly G-graded*.

Definition (Nystedt, Öinert 2017)

Let S be a G-graded ring. If, for every $g \in G$,

1. $S_g = S_g S_{g^{-1}} S_g$, and,
2. $S_g S_{g^{-1}}$ is an s-unital ideal of S_e,

then S is called *nearly epsilon-strongly G-graded*.

Nearly epsilon-strongly graded rings

Definition

A ring S is called *s-unital* if $x \in xS \cap Sx$ for all $x \in S$.

Examples:

1. Leavitt path algebras (Nystedt-Öinert, 2017)
   1. Finite graph $\implies$ epsilon-strongly $\mathbb{Z}$-graded.
   2. Any graph $\implies$ nearly epsilon-strongly $\mathbb{Z}$-graded.
2. unital partial crossed products (Nystedt-Öinert-Pinedo, 2016)
3. algebraic Cuntz-Pimsner rings (L., 2019)

Remark

Only unital rings admit epsilon-strong gradations. Only s-unital rings admit nearly epsilon-strong gradations.

Remark

Let S be a G-graded ring. Then the following implications hold

$$\text{unital strongly graded} \Rightarrow \text{epsilon strongly graded} \Rightarrow \text{nearly epsilon strongly graded}$$

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Algebraic analogues of graph C^*-algebra and a generalization of Leavitt algebras. (G. Abrams, G. Aranda Pino, P. Ara, M. A. Moreno, E. Pardo).

Definition

Let R be a ring and $E = (E^0, E^1, s, r)$ be a directed graph. The *Leavitt path algebra* attached to E with coefficients in R is the R-algebra generated by the symbols:

1. $\{v \mid v \in E^0\}$,
2. $\{f \mid f \in E^1\}$,
3. $\{f^* \mid f \in E^1\}$.

...

LPA definition (cont)

Definition

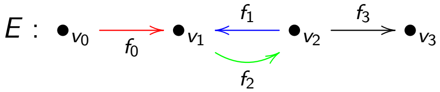
...

subject to the following relations:

1. $v_i v_j = \delta_{i,j} v_i$ for all $v_i, v_j \in E^0$,
2. $s(f)f = fr(f) = f$ and $r(f)f^* = f^*s(f) = f^*$ for all $f \in E^1$,
3. $f^*f' = \delta_{f,f'} r(f)$ for all $f, f' \in E^1$,
4. $\sum_{f \in E^1, s(f)=v} ff^* = v$ for all $v \in E^0$ for which $s^{-1}(v)$ is non-empty and finite.

Leavitt path algebras: Examples I

Ex: Consider the LPA associated with



Elements in $L_R(E)$:
$\alpha^* = f_1^* f_2^* f_0^* \in L_R(E)$
$v_0 \in L_R(E)$
$\gamma = f_0 \in L_R(E)$

$$\alpha^* \gamma = f_1^* f_2^* f_0^* f_0 = f_1^* f_2^* r(f_0) = f_1^* f_2^*.$$

Examples

Example

$$A_2 : \qquad \bullet_{v_1} \xrightarrow{f} \bullet_{v_2}$$

In this case, $L_R(A_2) \cong M_2(R)$.

Example

$$A_n : \qquad \bullet_{v_1} \xrightarrow{f_1} \bullet_{v_2} \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} \bullet v_n$$

In this case, $L_R(A_n) \cong M_n(R)$.

Examples

Example

R_1 :

In this case, $L_R(R_1) \cong R[x, x^{-1}]$.

Example

Take $n \geq 2$. Let R_n denote the rose with n petals graph having one vertex and n loops. Then, $L_K(R_n) \cong L_K(1, n)$ where $L_K(1, n)$ is the Leavitt algebra of type $(1, n)$.

Examples

The previous graphs have all been finite, but we also allow infinite graphs!

Example

Infinitely many vertices:

$$E' : \bullet_{v_1} \quad \bullet_{v_2} \quad \bullet_{v_3} \quad \bullet_{v_4} \quad \bullet_{v_5} \quad \bullet_{v_6} \quad \bullet_{v_7} \quad \bullet_{v_8} \quad \bullet_{v_9} \quad \bullet_{v_{10}} \cdots$$

In this case, $L_R(E') \cong \bigoplus_{i>0} Rv_i$.

Example

$$E'' : \bullet_{v_1} \xrightarrow{(\infty)} \bullet_{v_2}$$

The $\mathbb{Z}$-graded structure of Leavitt path algebras

Definition

The canonical $\mathbb{Z}$-grading of $L_R(E)$ is defined by,

$$\deg(\alpha\beta^*) = \operatorname{len}(\alpha) - \operatorname{len}(\beta).$$

Hazrat and Nystedt-Öinert showed the following:

$$\begin{array}{ccccc} E \text{ finite with no sinks} & \Rightarrow & E \text{ finite} & \Rightarrow & E \text{ a graph} \\ \Updownarrow & & \Updownarrow & & \Updownarrow \\ L_R(E) \text{ unital strong} & \Rightarrow & L_R(E)\ \epsilon\text{-strong} & \Rightarrow & L_R(E) \text{ nearly } \epsilon\text{-strong} \end{array}$$

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Connell's Theorem

Definition

A ring R is called prime if for all ideals A, B of R, $AB = 0$ implies that $A = 0$ or $B = 0$.

Theorem (Connell, 1963)

Let R be a unital ring and let G be a group. The group ring $R[G]$ is prime if and only if R is prime and G has no non-trivial finite normal subgroups.

Passman's Theorem

Definition (Passman, 1984)

Let S be a unital strongly G-graded ring. Then G acts on the ideals of S_e by $I^x := S_{x^{-1}} I S_x$. Let H be a subgroup of G. If $I^x = I$ for every $x \in H$, then I is called H-invariant.

Theorem (Passman, 1984)

Let S be a unital strongly G-graded ring. Then S is not prime if and only if there exist:

1. subgroups $N \lhd H \subseteq G$ with N finite;
2. an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and
3. nonzero H-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq I S_N$ and $\tilde{A}\tilde{B} = \{0\}$.

Invariant ideals

Updated definition:

Definition (L., Lundström, Wagner, Öinert, 2021, cf. Passmann)

Let S be a G-graded ring.

1. For a subset $I \subseteq S$ and $x \in G$, we consider the set $I^x := S_{x^{-1}} I S_x$.
2. Let H be a subgroup of G. We say that I is H-invariant if $I^x \subseteq I$ for every $x \in H$.
3. Let N be a normal subgroup of H. We say that I is H/N-invariant if $S_{C^{-1}} I S_C \subseteq I$ for every $C \in H/N$.

Not a group action! $(I^x)^y \neq I^{xy}$.

Our main result

Theorem (L., Lundström, Öinert, Wagner 2021)

Let S be a nearly epsilon-strongly G-graded ring. Then S is not prime if and only if there exist:

1. subgroups $N \lhd H \subseteq G$ with N finite;
2. an H-invariant ideal I of S_e such that $I^x I = \{0\}$ for every $x \in G \setminus H$, and
3. nonzero H/N-invariant ideals $\tilde{A}, \tilde{B}$ of S_N such that $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.

Remark

Let S be a unital strongly G-graded ring. "Passman"-invariant ideal coincide with invariant ideal ($I^x \subseteq I \iff I^x = I$). H/N-invariant implies H-invariant.

Torsion-free grading groups

Definition

A proper G-invariant ideal Q of S_e is called *G-prime* if for all G-invariant ideals A, B of S_e we have $A \subseteq Q$ or $B \subseteq Q$ whenever $AB \subseteq Q$. The ring S_e is called G-prime if $\{0\}$ is a G-prime ideal of S_e.

Theorem (L., Lundström, Öinert, Wagner, 2021)

Suppose that G is torsion-free and that S is nearly epsilon-strongly G-graded. Then S is prime if and only if S_e is G-prime.

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Application to Prime Leavitt path algebras

Definition

Let E be a directed graph. The graph E is said to satisfy Condition (MT-3) if for every pair of vertices $u, v \in E^0$, there is some vertex $w \in E^0$ such that there are paths (possibly of zero length) from u to w and from v to w. (Confluence vertex w).

Example

$$A_n : \qquad \bullet_{v_1} \xrightarrow{f_1} \bullet_{v_2} \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} \bullet v_n$$

Note that A_n satisfies (MT-3).

Application to Prime Leavitt path algebras

The following theorem generalizes work by Abrams-Bell-Rangaswamy and Larki:

Theorem (L., Lundström, Öinert, Wagner 2021)

Let $L_R(E)$ be a Leavitt path algebra over a unital ring R. Then $L_R(E)$ is prime if and only if R is prime and E satisfied Condition (MT-3).

Example

Take $E := A_n$ in the above theorem. Then $L_R(A_n) \cong M_n(R)$ is prime if and only if R is prime.

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Unital partial crossed products

Definition

A *unital twisted partial action of G on R* is a triple $(\{\alpha_g\}_{g\in G}, \{D_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G})$ where for each $g \in G$, the D_g's are unital ideals of R, $\alpha_g : D_{g^{-1}} \to D_g$ are ring isomorphisms and for each $(g, h) \in G \times G$, $w_{g,h}$ is an invertible element in $D_g D_{gh}$. For all $g, h \in G$:

(P1) $\alpha_e = \mathrm{id}_R$;

(P2) $\alpha_g(D_{g^{-1}} D_h) = D_g D_{gh}$;

(P3) if $r \in D_{h^{-1}} D_{(gh)^{-1}}$, then $\alpha_g(\alpha_h(r)) = w_{g,h}\alpha_{gh}(r)w_{g,h}^{-1}$;

(P4) $w_{e,g} = w_{g,e} = 1_g$;

(P5) if $r \in D_{g^{-1}} D_h D_{hl}$, then $\alpha_g(rw_{h,l})w_{g,hl} = \alpha_g(r)w_{g,h}w_{gh,l}$.

Definition

Given a unital twisted partial action of G on R, we can form the *unital partial crossed product* $R \star_{\alpha}^{w} G = \bigoplus_{g \in G} D_g \delta_g$ where the δ_g's are formal symbols. For $g, h \in G$, $r \in D_g$ and $r' \in D_h$ the multiplication is defined by the rule:

(P6) $(r\delta_g)(r'\delta_h) = r\alpha_g(r'1_{g^{-1}})w_{g,h}\delta_{gh}$.

$R \star_{\alpha}^{w} G$ is an associative ring with a natural epsilon-strong G-grading (Nystedt-Öinert-Pinedo).

Theorem (L., Lundström, Öinert, Wagner, 2021)

Suppose that G is torsion-free and that $R \star_{\alpha}^{w} G$ is a unital partial crossed product. Then $R \star_{\alpha}^{w} G$ is prime if and only if R is G-prime.

Theorem (L.,Lundström, Öinert, Wagner, 2021)

The unital partial crossed product $R \star_\alpha^w G$ is not prime if and only if there are:

1. *subgroups $N \lhd H \subseteq G$ with N finite,*
2. *an ideal I of R such that*
   - $\alpha_h(I1_{h^{-1}}) = I1_h$ *for every* $h \in H$,
   - $I \cdot \alpha_g(I1_{g^{-1}}) = \{0\}$ *for every* $g \in G \setminus H$, *and*
3. *nonzero ideals $\tilde{A}, \tilde{B}$ of $R \star_\alpha^w N$ such that $\tilde{A}, \tilde{B} \subseteq I \cdot (R \star_\alpha^w N)$ and $\tilde{A} \cdot 1_h \delta_h \cdot \tilde{B} = \{0\}$ for every $h \in H$.*

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Graded prime spectrum of Leavitt path algebras

Definition

A proper graded ideal P of S is called graded prime if for all graded ideals A, B of S, we have $A \subseteq P$ or $B \subseteq P$ whenever $AB \subseteq P$.

1. $\text{Spec}_\gamma(L_k(E))$ is used in the construction of the *algebraic filtered K-theory* by Eilers-Restorff-Ruiz-Sørensen, 2021.
2. $\text{Spec}_\gamma(L_k(E))$ can be described using so-called admissible pairs (I, H) (Larki).

Up next: an alternative description of $\text{Spec}_\gamma(S)$ for S being a general nearly epsilon-strongly G-graded rings.

The graded prime spectrum of nearly epsilon-strongly graded rings

Let S be a G-graded ring and let I be an ideal of S. Define $I_e := I \cap S_e$.

Theorem (L., Lundström, Öinert, Wagner 2021, cf. Nastasescu-van Oystaeyen)

Let S be nearly epsilon-strongly G-graded. The map $I \mapsto I_e$ is a bijection

$$\{\text{graded ideals of } S\} \leftrightarrow \{G - \text{invariant ideals of } S_e\}.$$

Definition

A proper G-invariant ideal Q of S_e is called G-prime if for all G-invariant ideals A, B of S_e , we have $A \subseteq Q$ or $B \subseteq Q$ whenever $AB \subseteq Q$.

The graded prime spectrum of nearly epsilon-strongly graded rings

Theorem (L., Lundström, Öinert, Wagner 2021, cf. Nastasescu-van Oystaeyen)

Let S be nearly epsilon-strongly G-graded. The map $I \mapsto I_e$ is a bijection

$$\{\text{graded prime ideals of } S\} \leftrightarrow \{G-\text{prime ideals of } S_e\}.$$

Question

Is this useful for developing an algebraic filtered K-theory for so-called algebraic Cuntz-Pimsner rings (Carlsen-Ortega), or general nearly epsilon-strongly G-graded rings?

Thank you for your attention!