Projective Real Calculi and the Levi-Civita connection

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Overview

Real Calculi, Definition

A real calculus is a structure $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$, where

- $\mathcal{A}$ is a unital *-algebra,
- $\mathfrak{g}$ is a real Lie algebra and $D : \mathfrak{g} \to \mathrm{Der}(\mathcal{A})$ is a faithful representation of $\mathfrak{g}$ as a set of hermitian derivations,
- M is a (right) $\mathcal{A}$-module, and
- $\varphi : \mathfrak{g} \to M$ is a $\mathbb{R}$-linear map such that $\varphi(\mathfrak{g})$ generates M.

Let Σ be a smooth manifold. With

- $\mathcal{A} = \mathcal{C}^\infty(\Sigma)$,
- $\mathfrak{g} = \mathrm{Der}(\mathcal{C}^\infty(\Sigma))$ and $D = \mathrm{id}_{\mathfrak{g}}$,
- $M = \mathfrak{X}(\Sigma)$ (the module of smooth vector fields over Σ), and
- $\varphi =$ the natural isomorphism between smooth vector fields and derivations,

we have that $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ is a real calculus.

Historical overview

- Introduced by J. Arnlind and M. Wilson in 2015, RC were used to discuss Riemannian curvature for NC 3-sphere and to develop a NC version of the Gauss-Bonnet theorem for the NC 4-sphere.
- RC were used by J. Arnlind and A. Tiger Norkvist to develop a theory of embeddings in NCG, and it was shown how the noncommutative torus could be minimally embedded into the noncommutative 3-sphere. In connection to this, RC homomorphisms were developed.
- Lately, RC were studied as algebraic objects using RC homomorphisms and, in particular, the connection between "free" and "projective" RC was studied.

Nontrivial example

Let $\mathcal{A} = \mathrm{Mat}_N(\mathbb{C})$, and let $\mathfrak{g} \subseteq \mathfrak{sl}_N(\mathbb{C})$ be a Lie algebra of skew-hermitian matrices with basis $\{\delta_1, ..., \delta_n\}$. Since every derivation of $\mathcal{A}$ is inner (i.e., they are of the form $\partial = [\delta, \cdot]$ for a unique $\delta \in \mathfrak{sl}_N(\mathbb{C})$, with ∂ being hermitian iff δ is skew-hermitian) we may take D to be the representation given by $D : \delta \mapsto [\delta, \cdot]$.

If we let $\tilde{M} = \mathcal{A}^n$, and let $\tilde{\varphi}$ be such that $\{\tilde{\varphi}(\delta_1), ..., \tilde{\varphi}(\delta_n)\}$ is a basis of $\mathcal{A}^n$, then $\tilde{C}_\mathcal{A} = (\mathcal{A}, \mathfrak{g}_D, \mathcal{A}^n, \tilde{\varphi})$ is a so-called free real calculus (i.e., $\tilde{M}$ is free, and any basis of $\mathfrak{g}$ generates a basis of $\tilde{M}$ through $\tilde{\varphi}$).

This can be used to generate real calculi $C_\mathcal{A} = (\mathcal{A}, \mathfrak{g}_D, P(\mathcal{A}^n), P \circ \tilde{\varphi})$ where $P : \mathcal{A}^n \to \mathcal{A}^n$ is a projection. A real calculus where M is projective is called projective.

The Connection between Free and Projective Real Calculi

Every real projective calculus $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ is isomorphic to a projective real calculus obtained as the "projection" of a free real calculus.

Proposition

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ be a real calculus, where M is projective and $\dim \mathfrak{g} = n$. Then there exists a free real calculus $\tilde{C}_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, \mathcal{A}^n, \tilde{\varphi})$ and a projection $P : \mathcal{A}^n \to \mathcal{A}^n$ such that $(\mathcal{A}, \mathfrak{g}_D, P(\mathcal{A}^n), P \circ \tilde{\varphi}) \simeq C_{\mathcal{A}}$.

Thus, we may develop a theory of projective real calculi by using the additional structure provided by a free real calculus.

Metrics

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ be a real calculus. A *metric* $h : M \times M \to \mathcal{A}$ is a Hermitian form that is non-degenerate, i.e.

- $h(m_1 + m_2, n) = h(m_1, n) + h(m_2, n)$ for all $m_1, m_2, n \in M$,
- $h(m, n \cdot a) = h(m, n)a$ for all $m, n \in M$, $a \in \mathcal{A}$,
- $h(m, n) = h(n, m)^*$ for all $m, n \in M$, and
- $h(m, n) = 0$ for all $n \in M \Rightarrow m = 0$.

Moreover, if $h(\varphi(\partial_1), \varphi(\partial_2)) = h(\varphi(\partial_1), \varphi(\partial_2))^*$ for all $\partial_1, \partial_2 \in \mathfrak{g}$ (i.e., it is truly symmetric on $\varphi(\mathfrak{g})$) then $(C_{\mathcal{A}}, h)$ is called a real metric calculus.

One may consider invertible metrics, i.e., metrics such that the map $\hat{h} : M \to M^*$, defined by $\hat{h}(m)(n) = h(m, n)$, is invertible.

Connections

Let $C_{\mathcal{A}} = (\mathcal{A}, \mathfrak{g}_D, M, \varphi)$ be a real calculus. An affine connection $\nabla : \mathfrak{g} \times M \to M$ is a map that satisfies

- $\nabla_\partial(m + n) = \nabla_\partial m + \nabla_\partial n$ for all $m, n \in M$ and $\partial \in \mathfrak{g}$,
- $\nabla_{\lambda\partial_1+\partial_2} m = \lambda\nabla_{\partial_1} m + \nabla_{\partial_2} m$ for all $m \in M$, $\lambda \in \mathbb{R}$ and $\partial_1, \partial_2 \in \mathfrak{g}$, and
- $\nabla_\partial(m \cdot a) = (\nabla_\partial m) \cdot a + m \cdot \partial(a)$ for all $m \in M$, $\partial \in \mathfrak{g}$ and $a \in \mathcal{A}$.

$(C_{\mathcal{A}}, h, \nabla)$ is a real connection calculus if

$$h(\nabla_{\partial_1}\varphi(\partial_2), \varphi(\partial_3)) = h(\nabla_{\partial_1}\varphi(\partial_2), \varphi(\partial_3))^*, \quad \partial_1, \partial_2, \partial_3 \in \mathfrak{g}.$$

∇ is compatible with the metric h if

$$\partial(h(m_1, m_2)) = h(\nabla_\partial m_1, m_2) + h(m_1, \nabla_\partial m_2), \quad m_1, m_2 \in M.$$

The role of φ

The map $\varphi : \mathfrak{g} \to M$ enables us to discuss the concept of torsion:

$$T(\varphi(\partial_1), \varphi(\partial_2)) = \nabla_{\partial_1}\varphi(\partial_2) - \nabla_{\partial_2}\varphi(\partial_1) - \varphi([\partial_1, \partial_2]);$$

and ∇ is said to be torsion-free if $T(\varphi(\partial_1), \varphi(\partial_2)) = 0$ for all $\partial_1, \partial_2 \in \mathfrak{g}$.
A real connection calculus $(C_{\mathcal{A}}, h, \nabla)$ is pseudo-Riemannian if ∇ is torsion-free and compatible with the metric.

Theorem

Given a real metric calculus $(C_{\mathcal{A}}, h)$, there is at most one connection ∇ such that $(C_{\mathcal{A}}, h, \nabla)$ is pseudo-Riemannian.

If $(C_{\mathcal{A}}, h, \nabla)$ is pseudo-Riemannian, then ∇ is called the Levi-Civita connection.

Existence of the Levi-Civita connection

Given a real metric calculus it is not guaranteed that the Levi-Civita connection exists. But if $(C_{\mathcal{A}}, h)$ is a real metric calculus where $C_{\mathcal{A}}$ is a free real calculus and the metric h is invertible, then the Levi-Civita connection exists.

For general projective real metric calculi the question becomes more interesting.

Return of the matrix example

Let $C_{\mathcal{A}} = (\mathrm{Mat}_N(\mathbb{C}), \mathfrak{g}_D, \mathbb{C}^N, \varphi)$, where $\mathfrak{g} = \langle \partial \rangle$ and $\partial(A) = [\hat{D}, A]$; since ∂ is a hermitian derivation we have that $\hat{D}$ is anti-hermitian. Moreover, we have that $\varphi(\partial) \neq 0$.

Every metric h on $\mathbb{C}^N$ can be written as $h(u, v) = x \cdot u^\dagger v$, where $x \neq 0$ is a real number and $\dagger$ denotes the hermitian conjugate; for every metric h on $\mathbb{C}^N$, $(C_{\mathcal{A}}, h)$ is a real metric calculus.

- Q: Does the Levi-Civita connection exist in this case?
- A: Yes, but only if $\varphi(\partial)$ is an eigenvector of $\hat{D}$.

Thus, the choice of $\varphi : \mathfrak{g} \to \mathbb{C}^N$ affects the existence of the Levi-Civita connection.

A more general case

Given a projective real metric calculus $(C_{\mathcal{A}}, h)$ where the metric is invertible, one can derive a purely algebraic condition for the existence of the Levi-Civita connection using the fact that every projective calculus is the projection of a free real calculus:

$$p_l^q \partial_i(p_j^l) = h^{qr} \Lambda_{r,ik}(\delta_j^k \mathbb{1} - p_j^k),$$

where p_j^i are the projection coefficients and the terms $\Lambda_{r,ik}$ are derived from a noncommutative analogue of Koszul's formula.

The main question

How to interpret this?

Topics of research

- Why does there sometimes NOT exist a LC connection at all?
  1. Problems with definitions?
  2. Problems with "unnatural" choice of map $\varphi : \mathfrak{g} \to M$?
- Suppose $\mathcal{A}$, $\mathfrak{g}$, M and h are given? Can we find a map $\varphi : \mathfrak{g} \to M$ such that a Levi-Civita connection exists?

The End