# On Solvability and Nilpotency of $(n+1)$-Hom-Lie algebras Induced by $n$-Hom-Lie algebras

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# Outline

- Introduction and context.
- Basic definitions and properties.
- Construction of $(n+1)$-Hom-Lie algebras induced by $n$-Hom-Lie algebras.
- Comparing solvability and nilpotency of an $n$-Hom-Lie algebra and those of an $(n+1)$-Hom-Lie algebra induced by it.

This talk is based on a work done in collaboration with Sergei Silvestrov and Abdenacer Makhlouf.

# Introduction and context

- Nambu, Y.: Generalized Hamiltonian dynamics (1973)
- Takhtajan, L. A.: On foundation of the generalized Nambu mechanics (1994)
- Filippov V. T., $n$-Lie algebras (1985)
- Kasymov Sh. M., Theory of $n$-Lie algebras (1987)
- Hartwig J. T., Larsson D., Silvestrov S. D., Deformations of Lie algebras using $\sigma$-derivations (2003)
- Larsson D., Silvestrov S., Quasi-hom-Lie algebras, central extensions and $2$-cocycle-like identities (2005)
- Makhlouf A., Silvestrov S. D., Hom-algebra structures (2006)
- Ataguema H., Makhlouf A., Silvestrov S., Generalization of n-ary Nambu algebras and beyond (2009)
- Yau D., On $n$-ary Hom-Nambu and Hom-Nambu-Lie algebras (2012)

# Introduction and context

- Awata, H., Li, M., Minic, D., Yoneya, T.: On the quantization of Nambu brackets (2001)
- Arnlind J., Makhlouf A., Silvestrov S., Ternary Hom-Nambu-Lie algebras induced by Hom-Lie algebras (2010)
- Arnlind J., Makhlouf A., Silvestrov S., Construction of $n$-Lie algebras and $n$-ary Hom-Nambu-Lie algebras (2011)
- Arnlind J., Kitouni A., Makhlouf A., Silvestrov S., Structure and Cohomology of $3$-Lie algebras induced by Lie algebras (2014)
- Kitouni A., Makhlouf A., Silvestrov S., On $(n+1)$-Hom-Lie algebras induced by $n$-Hom-Lie algebras (2016)
- Bai, R., Bai, C., Wang, J.: Realizations of 3-Lie algebras (2010)
- Bai, R., Wu, Y., Li, J., Zhou, H.: Constructing $(n+1)$-Lie algebras from $n$-Lie algebras (2012)
- Abramov V., Super 3-Lie algebras induced by super Lie algebras (2017)

All vector spaces are considered over a field of characteristic $0$.

### Definition

An $n$-Hom-Lie algebra is a vector space $A$ together with a skew-symmetric $n$-linear map $[\cdot, ..., \cdot]$ and $(n-1)$ linear maps $\alpha_i, 1 \leq i \leq n-1$ defined on $A$ satisfying the Hom-Nambu-Filippov identity:

$$[\alpha_1(x_1), ..., \alpha_{n-1}(x_{n-1}), [y_1, ..., y_n]] =$$
$$\sum_{i=1}^{n}[\alpha_1(y_1), ..., \alpha_{i-1}(y_{i-1}), [x_1, ..., x_{n-1}, y_i], \alpha_i(y_{i+1}), ..., \alpha_{n-1}(y_n)],$$
$$\forall x_1, ..., x_{n-1}, y_1, ..., y_n \in A.$$

## Definition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$, $(B, \{\cdot, ..., \cdot\}, \beta_1, ..., \beta_{n-1})$ be $n$-Hom-Lie algebras. An $n$-Hom-Lie algebra morphism is a linear map $f : A \to B$ satisfying the conditions:

- $f([x_1, ..., x_n]) = \{f(x_1), ..., f(x_n)\}$, for all $x_1, ..., x_n \in A$.
- $f \circ \alpha_i = \beta_i \circ f$, for all $i : 1 \leq i \leq n - 1$.

A linear map satisfying only the first condition is called a weak morphism.

## Definition

We refer to an $n$-Hom-Lie algebra $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ such that $\alpha_1 = \alpha_2 = .... = \alpha_{n-1} = \alpha$ by $(A, [\cdot, ..., \cdot], \alpha)$.

- It is said to be multiplicative if $\alpha$ is an algebra morphism.
- It is said to be regular if it is multiplicative and $\alpha$ is an isomorphism.

### Definition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ be an $n$-Hom-Lie algebra. Let $B$ be a subspace of $A$ invariant under all the linear maps $\alpha_i$:

- If for all $x_1, ..., x_n \in B$ we have $[x_1, ..., x_n] \in B$, then $B$ is a subalgebra of $A$.
- If for all $x_1, ..., x_{n-1} \in A$, and $y \in B$ we have $[x_1, ..., x_{n-1}, y] \in B$, then $B$ is an ideal of $A$.

If we drop the invariance under the twisting maps, $B$ will be called a weak subalgebra or a weak ideal respectively.

# Solvability and nilpotency of $n$-Hom-Lie algebras

## Definition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ be an $n$-Hom-Lie algebra, and let $I$ be an ideal of $A$. For $2 \leq k \leq n$, we define the $k$-derived series of the ideal $I$ by:

$$D_k^0(I) = I \text{ and } D_k^{p+1}(I) = \left[\underbrace{D_k^p(I), ..., D_k^p(I)}_{k}, \underbrace{A, ..., A}_{n-k}\right].$$

We define the $k$-central descending series of $I$ by:

$$C_k^0(I) = I \text{ and } C_k^{p+1}(I) = \left[C_k^p(I), \underbrace{I, ..., I}_{k-1}, \underbrace{A, ..., A}_{n-k}\right].$$

If there exists $r \in \mathbb{N}$ such that $D_k^r(I) = \{0\}$ (resp. $C_k^r(I) = \{0\}$), the ideal $I$ is said to be $k$-solvable (resp. $k$-nilpotent).

# $(n+1)$-Hom-Lie algebras induced by $n$-Hom-Lie algebras

## Definition

Let $A$ be a vector space. For an $n$-linear map $\phi : A^n \to A$ we call a linear map $\tau : A \to \mathbb{K}$ a *$\phi$-trace* if $\tau(\phi(x_1, ..., x_n)) = 0$ for all $x_1, ..., x_n \in A$.

Let $(A, \phi, \alpha_1, ..., \alpha_{n-1})$ be an $n$-Hom-Lie algebra, $\tau$ a $\phi$-trace and $\alpha_n : A \to A$ a linear map. Define $\phi_\tau : A^{n+1} \to A$ by:

$$\phi_\tau(x_1, ..., x_{n+1}) = \sum_{k=1}^{n+1} (-1)^{k-1} \tau(x_k) \phi(x_1, ..., x_{k-1}, x_{k+1}, ..., x_{n+1})$$

# $(n+1)$-Hom-Lie algebras induced by $n$-Hom-Lie algebras

**Theorem**

*If it holds that*

$$\tau(\alpha_i(x))\,\tau(y) = \tau(x)\tau(\alpha_i(y)) \quad (1)$$

$$\tau(\alpha_i(x))\,\alpha_j(y) = \tau(\alpha_j(x))\,\alpha_i(y) \quad (2)$$

*for all $i, j \in \{1, ..., n\}$ and all $x, y \in A$, then $(A, \phi_\tau, \alpha_1, ..., \alpha_n)$ is an $(n+1)$-Hom-Lie algebra. We shall say that $(A, \phi_\tau, \alpha_1, ..., \alpha_n)$ is induced by $(A, \phi, \alpha_1, ..., \alpha_{n-1})$. We refer to $A$ when considering the given $n$-Hom-Lie algebra and $A_\tau$ when considering the induced $(n+1)$-Hom-Lie algebra.*

# $(n+1)$-Hom-Lie algebras induced by $n$-Hom-Lie algebras

The condition of the existence of an element $u \in A$ satisfying

$$\forall x_1, ..., x_n \in A, [x_1, ..., x_n, u]_\tau = [x_1, ..., x_n], \tag{3}$$

appears often and allows to have more properties for the induced algebra. It was used for several results before, and will be used bellow. Such an element is characterized by:

**Proposition**

*An element $u \in A$, where the algebra $A$ is not abelian, satisfies*

$$[x_1, ..., x_n, u]_\tau = [x_1, ..., x_n], \forall x_1, ..., x_n \in A,$$

*if and only if $u \in Z(A)$ and $\tau(u) = (-1)^n$.*

# $(n+1)$-Hom-Lie algebras induced by $n$-Hom-Lie algebras

**Proposition**

*Let $B$ be a subalgebra of $A$. If $\alpha_n(B) \subseteq B$ then $B$ is also a subalgebra of $A_\tau$.*

**Proposition**

*Let $J$ be an ideal of $A$. If $\alpha_n(J) \subseteq J$, then $J$ is an ideal of $A_\tau$ if and only if*

$$[A, ..., A] \subseteq J \text{ or } J \subseteq \ker \tau.$$

**Proposition**

*If there exists $u \in A$ satisfying condition (3), then every ideal of $A_\tau$ is an ideal of $A$.*

Let $(A, [\cdot, ..., \cdot], \alpha)$ be a multiplicative $n$-Hom-Lie algebra, $\tau$ be a trace satisfying $\tau \circ \alpha = \tau$, and $(A, [\cdot, ..., \cdot]_\tau, \alpha)$ the induced algebra.

## Proposition

*Let $(C^p(A))_p$ be the central descending series of $A$, and $(C^p(A_\tau))_p$ be the central descending series of $A_\tau$. Then we have*

$$C^p(A_\tau) \subset C^p(A), \forall p \in \mathbb{N}.$$

*If there exists $u \in A$ such that $[u, x_1, ..., x_n]_\tau = [x_1, ..., x_n], \forall x_1, ..., x_n \in A$, then*

$$C^p(A_\tau) = C^p(A), \forall p \in \mathbb{N}.$$

## Theorem

*if $A$ is nilpotent of class $p$, we have $A_\tau$ is nilpotent of class at most $p$. Moreover, if there exists $u \in A$ such that $[u, x_1, ..., x_n]_\tau = [x_1, ..., x_n], \forall x_1, ..., x_n \in A$, then $A$ is nilpotent of class $p$ if and only if $A_\tau$ is nilpotent of class $p$.*

In the following, let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ be an $n$-Hom-Lie algebra, $\tau$ and $\alpha_n$ be a trace and a linear map satisfying the compatibility conditions, and $(A, [\cdot, ..., \cdot]_\tau, \alpha_1, ..., \alpha_n)$ be the induced algebra, let $I$ be an ideal of $A$ that is also an ideal of $A_\tau$. We denote by $(D_k^r(I_\tau))$ and $(C_k^r(I_\tau))$ the $k$-derived series and the $k$-central descending series of an ideal $I$ in the induced algebra.

## Proposition

*For all $2 \leq k \leq n$ and $r \in \mathbb{N}$, we have:*

$$D_k^r(I_\tau) \subseteq D_k^r(I),$$

*and if there exists $u \in A$ such that*

$$\forall x_1, ..., x_n \in A, [x_1, .., x_n, u]_\tau = [x_1, ..., x_n],$$

*then $D_k^r(I_\tau) = D_k^r(I)$.*

**Proposition**

*Let $I$ be an ideal of $A$ that is also an ideal of $A_\tau$, then for all $2 \leq k \leq n$. If $I$ is $k$-solvable of class $r$ in $A$ then it is $k$-solvable of class at most $r$ in $A_\tau$. Moreover if there exists $u \in A$ satisfying condition (3), then the converse also holds.*

## Proposition

*Let $I$ be an ideal of $A$ that is also an ideal of $A_\tau$, then for all $2 < k \leq n+1$ and $r \in \mathbb{N}$, we have:*

$$C_k^r(I_\tau) \subseteq C_{k-1}^r(I).$$

*Moreover, if there exists $u \in A$ such that for all $x_1, ..., x_n \in A$, we have $[x_1, .., x_n, u]_\tau = [x_1, ..., x_n]$, then, for all $2 \leq k \leq n$, we have:*

$$C_k^r(I) \subseteq C_k^r(I_\tau).$$

## Proposition

*Let $I$ be an ideal of $A$ that is also an ideal of $A_\tau$ and suppose that there exists $u \in A$ such that for all $x_1, ..., x_n \in A$, we have $[x_1, .., x_n, u]_\tau = [x_1, ..., x_n]$ and that this $u$ is an element of $I$, then for all $2 < k \leq n+1$ and $r \in \mathbb{N}$, we have:*

$$C_k^r(I_\tau) = C_{k-1}^r(I).$$

## Proposition

*Let $I$ be an ideal of $A$ such that $I \subseteq \ker(\tau)$, then for all $2 \leq k \leq n$ and $r \in \mathbb{N}$, we have:*

$$C_k^r(I_\tau) \subseteq C_k^r(I).$$

*Moreover, if there exists $u \in A$ such that for all $x_1, ..., x_n \in A$, we have $[x_1, .., x_n, u]_\tau = [x_1, ..., x_n]$, then we have:*

$$C_k^r(I) = C_k^r(I_\tau).$$

**Theorem**

*Let $I$ be an ideal of $A$ that is also an ideal of $A_\tau$, then for all $2 \leq k \leq n$:*

1. *If $I$ is $k$-nilpotent of class $r$ in $A$ then it is $(k+1)$-nilpotent of class at most $r$ in $A_\tau$. If there exists $u \in A$ satisfying condition (3) and that $u \in I$, then the converse is true.*
2. *If $I$ is $k$-nilpotent of class $r$ in $A_\tau$ and there exists $u \in A$ satisfying condition (3) then it is $k$-nilpotent of class at most $r$ in $A$.*
3. *If $I$ is $k$-nilpotent of class $r$ in $A$ and $I \subseteq \ker \tau$ then it is $k$-nilpotent of class at most $r$ in $A_\tau$.*

# Examples

Consider the 5-dimensional 3-Hom-Lie algebra $(A, [\cdot,\cdot,\cdot], \alpha)$, defined with respect to the basis $(e_i)_{1\leq i\leq 5}$ by:

$$[e_2,e_3,e_4] = e_2 + \sqrt{2}e_3; [e_1,e_3,e_4] = \sqrt{2}e_2 + e_3; [e_1,e_2,e_4] = -e_1.$$

$$[\alpha] = \begin{pmatrix} 0 & 0 & -1 & a_{14} & 0 \\ 1 & \sqrt{2} & 0 & a_{24} & 0 \\ \sqrt{2} & 1 & 0 & a_{34} & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We choose the following trace map

$$\tau(x) = \tau\left(\sum_{k=1}^{5} x_i e_i\right) = t_4 x_4 + x_5,$$

The compatibility conditions are satisfied and we get the following 4-Hom-Lie algebra

$$[e_2,e_3,e_4,e_5]_\tau = e_2+\sqrt{2}e_3; [e_1,e_3,e_4,e_5]_\tau = \sqrt{2}e_2+e_3; [e_1,e_2,e_4,e_5]_\tau = -e_1$$

$$\ker \tau = \langle \{-e_4 + t_4 e_5, e_1, e_2, e_3\} \rangle$$

In this case, there exists an element $u$ satisfying the condition 3.

Table: Derived series for $A$

| | $r = 1$ | | $r \geq 2$ | |
|---|---|---|---|---|
| | $D_k^r(A)$ | $D_k^r(A_\tau)$ | $D_k^r(A)$ | $D_k^r(A_\tau)$ |
| $k = 2$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\langle \{e_1, e_2, e_3\} \rangle$ |
| $k = 3$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\{0\}$ | $\{0\}$ |
| $k = 4$ | N/A | $\langle \{e_1, e_2, e_3\} \rangle$ | N/A | $\{0\}$ |

Table: Derived series for $I = \ker \tau$

| | $r = 1$ | | $r \geq 2$ | |
|---|---|---|---|---|
| | $D_k^r(I)$ | $D_k^r(I_\tau)$ | $D_k^r(I)$ | $D_k^r(I_\tau)$ |
| $k = 2$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\langle \{e_1, e_2, e_3\} \rangle$ |
| $k = 3$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\langle \{e_1, e_2, e_3\} \rangle$ | $\{0\}$ | $\{0\}$ |
| $k = 4$ | N/A | $\{0\}$ | N/A | $\{0\}$ |

Table: Central descending series for $A$ and $I = \ker \tau$

| $r \geq 1$ | $C_k^r(A)$ | $C_k^r(A_\tau)$ |
|---|---|---|
| $k = 2$ | $\langle\{e_1, e_2, e_3\}\rangle$ | $\langle\{e_1, e_2, e_3\}\rangle$ |
| $k = 3$ | $\langle\{e_1, e_2, e_3\}\rangle$ | $\langle\{e_1, e_2, e_3\}\rangle$ |
| $k = 4$ | N/A | $\langle\{e_1, e_2, e_3\}\rangle$ |

| $r \geq 1$ | $C_k^r(I)$ | $C_k^r(I_\tau)$ |
|---|---|---|
| $k = 2$ | $\langle\{e_1, e_2, e_3\}\rangle$ | $\langle\{e_1, e_2, e_3\}\rangle$ |
| $k = 3$ | $\langle\{e_1, e_2, e_3\}\rangle$ | $\langle\{e_1, e_2, e_3\}\rangle$ |
| $k = 4$ | N/A | $\{0\}$ |

Consider the $5$-dimensional $3$-Hom-Lie algebra $(A, [\cdot,\cdot,\cdot], \alpha)$, defined with respect to the basis $(e_i)_{1\leq i\leq 5}$ by:

$$[e_2,e_3,e_4] = -e_1 - e_2; [e_1,e_3,e_4] = e_1; [e_2,e_4,e_5] = e_1; [e_1,e_4,e_5] = -e_2.$$

$$[\alpha] = \begin{pmatrix} 0 & 1 & 0 & a_{14} & 0 \\ -1 & 0 & 0 & a_{24} & 0 \\ 0 & 0 & 1 & a_{34} & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & a_{54} & -1 \end{pmatrix}.$$

We choose the following trace map

$$\tau(x) = \tau\left(\sum_{k=1}^{5} x_i e_i\right) = t_3x_3 + \frac{1}{2}a_{34}t_3x_4,$$

and we get the following $4$-Hom-Lie algebra

$$[e_1,e_3,e_4,e_5]_\tau = t_3e_2; [e_2,e_3,e_4,e_5]_\tau = -t_3e_1.$$

$$\ker\tau = \langle\{e_1, e_2, -\frac{1}{2}a_{34}e_3 + e_4, e_4\}\rangle$$

Note that in this case, there exists no element $u$ satisfying the condition 3.

Table: Derived series for $A$

| | $r=1$ | | $r \geq 2$ | |
|---|---|---|---|---|
| | $D_k^r(A)$ | $D_k^r(A_\tau)$ | $D_k^r(A)$ | $D_k^r(A_\tau)$ |
| $k=2$ | $\langle\{e_1, e_2\}\rangle$ | $\langle\{e_1, e_2\}\rangle$ | $\{0\}$ | $\{0\}$ |
| $k=3$ | $\langle\{e_1, e_2\}\rangle$ | $\langle\{e_1, e_2\}\rangle$ | $\{0\}$ | $\{0\}$ |
| $k=4$ | N/A | $\langle\{e_1, e_2\}\rangle$ | N/A | $\{0\}$ |

Table: Derived series for $I = \ker\tau$

| | $r=1$ | | $r \geq 2$ | |
|---|---|---|---|---|
| | $D_k^r(I)$ | $D_k^r(I_\tau)$ | $D_k^r(I)$ | $D_k^r(I_\tau)$ |
| $k=2$ | $\langle\{e_1, e_2\}\rangle$ | $\langle\{e_1, e_2\}\rangle$ | $\{0\}$ | $\{0\}$ |
| $k=3$ | $\langle\{e_1, e_2\}\rangle$ | $\langle\{e_1, e_2\}\rangle$ | $\{0\}$ | $\{0\}$ |
| $k=4$ | N/A | $\{0\}$ | N/A | $\{0\}$ |

Table: Central descending series for $A$ and $I = \ker \tau$

| $r \geq 1$ | $C_k^r(A)$ | $C_k^r(A_\tau)$ |
|---|---|---|
| $k=2$ | $\langle\{e_1, e_2\}\rangle$ | $\langle\{e_1, e_2\}\rangle$ |
| $k=3$ | $\langle\{e_1, e_2\}\rangle$ | $\langle\{e_1, e_2\}\rangle$ |
| $k=4$ | N/A | $\langle\{e_1, e_2\}\rangle$ |

| $r \geq 1$ | $C_k^r(I)$ | $C_k^r(I_\tau)$ |
|---|---|---|
| $k=2$ | $\langle\{e_1, e_2\}\rangle$ | $\langle\{e_1, e_2\}\rangle$ |
| $k=3$ | $\langle\{e_1, e_2\}\rangle$ | $\langle\{e_1, e_2\}\rangle$ |
| $k=4$ | N/A | $\{0\}$ |

# The End

Thank you