Dr. Jouko Mickelsson Or: How I Learned To Stop Worrying And Love The Gerbe

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1. Preface: Jouko Mickelsson

Figure: Jouko Mickelsson. Prof. emer., Department of Mathematics and Statistics, University of Helsinki.

Research interests include: current algebras, extensions of infinite dimensional Lie groups, twisted K-theory, and gerbes.

1. Preface: Fine arts

Figure: Renoir's painting "Petite fille á la gerbe"

1. Preface: Outline of the talk

1. What is a gerbe?
2. Applications of gerbes
3. Quantum gerbes

2. What is a gerbe?

We are basically in gerbe territory (for a smooth manifold X) if any one of the following is being considered:

- a cohomology class in $H^3(X, \mathbb{Z})$ (Dixmier and Douady)
- a codimension three submanifold $M^{n-3} \subseteq X^n$
- a Čech cocycle $[g_{\alpha\beta\gamma}] \in \check{H}^2(X, \underline{\mathbb{S}^1})$

In the last case, this means a 2-cocycle for the sheaf of germs of C^∞-functions with values in the circle.

2. What is a gerbe?

To understand gerbes, we need to consider the other objects in a hierarchy to which gerbes belong, and here the lowest form of life consists of circle-valued functions $f : X \to \mathbb{S}^1$. We have:

- a cohomology class in $H^1(X, \mathbb{Z})$
- a codimension one submanifold $M^{n-1} \subseteq X^n$
- a Čech cocycle $[g_\alpha] \in \check{H}^0(X, \underline{\mathbb{S}^1})$

2. What is a gerbe?

- The cohomology class is just the pull-back $f^*(x)$ of the generator $x \in H^1(\mathbb{S}^1, \mathbb{Z}) \cong \mathbb{Z}$.
- If we take the inverse image $f^{-1}(c)$ of a regular value $c \in \mathbb{S}^1$, then this is a codimension one submanifold of X.
- Finally, given an open covering $\{U_\alpha\}$ of X, a global function $f : X \to \mathbb{S}^1$ is built up out of local functions $g_\alpha : U_\alpha \to \mathbb{S}^1$ with

$$g_\beta g_\alpha^{-1} = 1 \quad \text{on} \quad U_\alpha \cap U_\beta.$$

Thus the three aspects of gerbes that we identified above all occur here, but in degree 1 rather than 3.

The next stage up in the hierarchy consists of a unitary line bundle L, or its principal $\mathbb{S}^1$-bundle of unitary frames. Here we have:

- a cohomology class in $H^2(X, \mathbb{Z})$ (Weil and Kostant)
- a codimension one submanifold $M^{n-2} \subseteq X^n$
- a Čech cocycle $[g_\alpha] \in \check{H}^1(X, \underline{\mathbb{S}^1})$

2. What is a gerbe?

- The degree 2 cohomology class is the first Chern class $c_1(L)$.
- If we take a smooth section of L with nondegenerate zeros, then this vanishes on X at a codimension two submanifold M.
- And if we take an open covering $\{U_\alpha\}$ over each set of which L is trivial, we have transition functions $g_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{S}^1$ which satisfy $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and the cocycle condition

$$g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1 \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma.$$

Thus the three aspects of gerbes that we identified above also all occur here, but in degree 2 rather than 3.

2. What is a gerbe?

Summarizing, we have:

$$
\begin{array}{lcl}
[\text{functions} \quad f : X \to \mathbb{S}^1] & \triangleq & H^1(X,\mathbb{Z}) \\
[\text{line bundles over} \quad X\ ] & \triangleq & H^2(X,\mathbb{Z}) \\
[\qquad ??? \qquad] & \triangleq & H^3(X,\mathbb{Z})
\end{array}
$$

Definition

A *gerbe* is a geometric realization of an element in $H^3(X,\mathbb{Z})$.

3. Geometric realization of $H^3(X, \mathbb{Z})$

Now let

- $\mathcal{H}$ be an infinite-dimensional Hilbert space,
- $\mathcal{U} = \mathcal{U}(\mathcal{H})$ be the Lie group of unitary operators on $\mathcal{H}$,
- $\mathcal{K} = \mathcal{K}(\mathcal{H})$ be the algebra of compact operators on $\mathcal{H}$,
- $G := \mathrm{Aut}(\mathcal{K})$ be the Lie group of automorphisms of $\mathcal{K}$,

and note that we have an exact sequence of Lie groups:

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow \mathcal{U} \longrightarrow G \longrightarrow 1. \tag{1}$$

3. Geometric realization of $H^3(X, \mathbb{Z})$

The following result is the basic reason behind our approach:

Theorem (Dixmier, Douady, Kuiper)

The Lie group $\mathcal{U}$ of unitary operators on $\mathcal{H}$ is contractible.

Note that $\mathcal{U}(1) \cong \mathbb{S}^1$ is *not* contractible!

Corollary (Dixmier, Douady)

We have natural bijections

$$\check{H}^1(X, \underline{G}) \cong \check{H}^2(X, \underline{\mathbb{S}^1}) \cong H^3(X, \mathbb{Z}).$$

3. Geometric realization of $H^3(X,\mathbb{Z})$

Recall that $G := \mathrm{Aut}(\mathcal{K})$.

Theorem

The Čech cohomology group $\check{H}^1(X,\underline{G})$ is in a natural bijection with the set of isomorphism classes of

- *locally trivial principal G-bundles $P \to X$,*
- *locally trivial algebra bundles $\mathbb{A} \to X$ with fibre $\mathcal{K}$ (aka. continuous-trace C^*-algebras),*
- *locally trivial projective Hilbert space bundles $\mathbb{P} \to X$.*

Hence, a gerbe (over X) is any of the mathematical objects described above. Just pick your favorite setting.

4. Why gerbes? Gerbes and quantization in field theory

- For a simple 1-connected compact Lie group G the generator of $H^3(G,\mathbb{Z}) \cong \mathbb{Z}$ is what gives rise to the whole theory of *loop groups*, their central extensions, and their representations.
- Let PG be the space of all smooth paths γ in G starting from $\gamma(0) = 1_G$ and with an arbitrary endpoint $\gamma(1) \in G$.
- $PG \to G$, $\gamma \mapsto \gamma(1_G)$ defines a principal bundle over G with fiber equal to the group ΩG of based loops in G.
- Assume that $\phi : \Omega G \to PU(\mathcal{H})$ is a projective representation of ΩG on some Hilbert space $\mathcal{H}$.
- Form the projective Hilbert space bundle $PG \times_{\Omega G} P(\mathcal{H})$. This is how gerbes appear in canonical quantization in field theory.

6. Why gerbes? Gerbes and T-duality

- The mathematical description of T-duality is based on pairs (P, δ) consisting of a *principal torus bundle* P (over a fixed manifold X) and a class $\delta \in H^3(P, \mathbb{Z})$ (often called *H-flux*).
- T-duality itself provides an involution

$$(P, \delta) \mapsto (P^{\#}, \delta^{\#})$$

of principal torus bundles with H-flux, satisfying a number of interesting properties (cf. Rosenberg's CBMS memoir).

6. Gerbes and T-duality

- It is known that each principal $\mathbb{S}^1$-bundle P with H-flux δ has a classical T-dual $(P^{\#}, \delta^{\#})$. However, this becomes false for general principal torus bundles.
- The issue may be resolved by passing over to quantum spaces. Indeed, it turns out that every principal $\mathbb{T}^2$-bundle with H-flux admits a "non-classical" T-dual, that is, a C^*-algebraic bundle of quantum 2-tori (*noncommutative principal* $\mathbb{T}^2$*-bundle*).

7. Towards a theory of quantum gerbes: The how

- Noncommutative principal bundles provide a natural framework for quantum gerbes. However, the C^*-algebraic setting is no longer available, as the structure group is infinite-dimensional.
- It also seems to be instructive to consider at first the case in which the "base space" is a quantum group, e. g. $SU_q(2)$.
- Another point of view is to study quantum gerbes as a system of quantum line bundles (i. e. invertible bimodules) which obey a certain cocycle condition.

7. Towards a theory of quantum gerbes: The why

- One goal is to attach a *quantum H-flux* to the quantum gerbe, e. g., an element in cyclic homology (geometric realization).
- Another goal is to investigate T-duality in the noncommutative setting of torus bundles and gerbes.

Thank you for your attention!