When is a group ring a Köthe ring?

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BTH

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Outline

"The basis theorem" from Linear Algebra

Theorem

Every (non-zero) vector space has a basis. (Using the axiom of choice.)

Remark

Fix a field $\mathbb{F}$. Let V be **any** vector space over $\mathbb{F}$.

- Then
$$V = \oplus_{i \in I} \mathbb{F} e_i$$
with I possibly infinite.
- $\mathbb{F} e_i$ is a *cyclic* $\mathbb{F}$-module, for every $i \in I$.
- Conclusion: V is a direct sum of cyclic modules!

Question

Let us replace $\mathbb{F}$ with a unital ring and consider modules over that ring. Could we have the above properties for those modules?

Köthe rings

Definition

- A unital ring S is said to be a *left (resp. right) Köthe ring* if every left (resp. right) S-module is a direct sum of cyclic modules.
- If S is both a left and a right Köthe ring, then S is simply called a *Köthe ring*.

Definition

A unital ring S is said to be an *artinian principal ideal ring* if it is

- left artinian (i.e. S satisfies the DCC on left ideals),
- right artinian (i.e. S satisfies the DCC on right ideals),
- a principal left ideal ring (i.e. whenever I is a left ideal of S, we have $I = Sx$ for some $x \in S$),
- and a principal right ideal ring (i.e. whenever I is a right ideal of S, we have $I = yS$ for some $y \in S$).

Köthe rings, continued

Theorem (Köthe, 1935)

Let S be a unital ring. If S is a left (resp. right) artinian principal ideal ring, then S is a left (resp. right) Köthe ring.

Theorem (Cohen & Kaplansky, 1951)

Let S be a unital commutative ring. Then S is a Köthe ring if and only if S is an artinian principal ideal ring.

Theorem (Faith & Walker, 1967)

Let S be a unital ring. If S is a left (resp. right) Köthe ring, then S is a left (resp. right) artinian ring.

Example (Nakayama, 1940)

There is a Köthe ring which is not a principal ideal ring.

The group ring $R[G]$

- Ingredients: a unital ring R and a group G
- $R[G]$ is a free left (and right) R-module with G as its basis; Every element is of the form $r_1 g_1 + r_2 g_2 + \ldots + r_n g_n$.
- Multiplication is defined by

$$(r_1 g_1)(r_2 g_2) = (r_1 r_2)(g_1 g_2)$$

for $r_1, r_2 \in R$ and $g_1, g_2 \in G$.

Question

Given a ring R and a group G, when is the group ring $R[G]$ a Köthe ring?

Artinianity of $R[G]$

Theorem (Connell, 1963)

Let R be a unital ring and let G be a group. The group ring $R[G]$ is left (resp. right) artinian if and only if R is left (resp. right) artinian and G is finite.

Principal ideal group rings

Remark

- G is $\mathcal{A}$-by-$\mathcal{B}$ (with $\mathcal{A}$, $\mathcal{B}$ two classes of groups) if there exists a normal subgroup $N \trianglelefteq G$, such that $N \in \mathcal{A}$ and $G/N \in \mathcal{B}$.
- A finite group G is a p-group if $|G| = p^k$ for some natural number k.
- A finite group G is a p'-group if $|G|$ is relatively prime to p.

Theorem (Passman 1977, Dorsey 2007)

Let K be a division ring, let G be a finite group and consider the group ring $K[G]$. The following two assertions are equivalent:

1. $K[G]$ *is a principal ideal ring.*
2. $\mathrm{char}(K) = 0$, *or*
   $\mathrm{char}(K) = p > 0$ *and G is p'-by-cyclic p.*

That is: $\exists N \trianglelefteq G$ such that G/N is cyclic, $(|N|, p) = 1$ and $|G/N| = p^k$.

Necessary conditions

Proposition (Baghdari & Öinert)

Let R be a unital ring and let G be a group. If the group ring $R[G]$ is a left (resp. right) Köthe ring, then the following assertions hold:

1. (i) R *is a left (resp. right) Köthe ring;*
2. (ii) G *is a finite group;*
3. (iii) $(R/I)[G]$ *is a left (resp. right) Köthe ring for every proper ideal* I *of* R*;*
4. (iv) $R[G/N]$ *is a left (resp. right) Köthe ring for every normal subgroup* N *of* G*;*
5. (v) $(R/I)[G/N]$ *is a left (resp. right) Köthe ring for every proper ideal* I *of* R *and every normal subgroup* N *of* G*.*

A result for non-commutative group rings

Theorem

Let R be a division ring with $\mathrm{char}(R) = 0$ and let G be a group. The group ring $R[G]$ is a Köthe ring if and only if G is a finite group.

Example

Consider $R[G]$ with

- $R = \mathbb{H}$, the quaternions, and
- $G = S_3$, the symmetric group on 3 letters.

Then $R[G]$ is a Köthe ring.

Two nice little lemmas

Lemma (Baghdari & Öinert)

Let R be a commutative unital ring and let $n > 1$ be an integer. The following statements are equivalent:

1. $n \cdot 1_R$ *is not invertible in* R*;*
2. *there is a prime divisor* q *of* n*, and a proper ideal* M *of* R *such that* R/M *is an integral domain with* $\mathrm{char}(R/M) = q$*.*

Lemma (Baghdari & Öinert)

Let (R, M) be a unital commutative local ring with $\mathrm{char}(R/M) = p$. Then for $n > 1$ we have that $n \cdot 1_R$ is not invertible in R if and only if p divides n.

Local coefficient rings

Theorem (Baghdari & Öinert)

Let (R, M) be a unital commutative local ring and let G be an abelian group. The following two assertions are equivalent:

1. *The group ring $R[G]$ is a Köthe ring;*
2. $\mathrm{char}(R/M) = 0$: *The ring R is a Köthe ring and G is a finite group. If R is not a field, then $|G| \cdot 1_R \in U(R)$.*
   $\mathrm{char}(R/M) = p > 0$: *The ring R is a Köthe ring and G is a finite p'-by-cyclic p group. If R is not a field, then $|G| \cdot 1_R \in U(R)$.*

Remark

(i)⇒(ii): $\mathrm{char}(R/M) = 0$ - quite easy! $\mathrm{char}(R/M) = p$ - needs a combination of results of Cohen & Kaplansky, Passman and Dorsey.
(ii)⇒(i): R a field - works by combining results of Connell, Passman and Köthe. R not a field - more tricky! Through pure projective modules and a result of Girvan (1973).

The main result (so far!)

Remark

Let R be a commutative unital artinian ring. There is a unique positive integer n, and local commutative unital artinian rings $R_1, \ldots, R_n$ such that $R \cong R_1 \times \ldots \times R_n$.

Theorem (Baghdari & Öinert)

Let R be a unital commutative ring and let G be an abelian group. The following two assertions are equivalent:

1. *The group ring $R[G]$ is a Köthe ring.*
2. *The ring R is Köthe, G is a finite group and G is p'-by-cyclic p, for every $p \in \pi = \{q \mid q = \mathrm{char}(R/M) \text{ for some } M \in \mathrm{Max}(R)\}$. Moreover, $|G| \cdot 1_{R_i} \in U(R_i)$ whenever R_i is a non-semiprimitive ring which appears in the decomposition of R.*

The end

THANK YOU FOR YOUR ATTENTION!