Non-commutative Henselian Rings

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SNAG, 2020

Content

Hensel's lemma and Henselian rings

Let (A, m) be a local ring, $k = A/m$ be the residue field of A and $\pi : A \to k$ be the quotient map. Suppose that $F(x) \in A[x]$ is a monic polynomial for which we have a factorization

$$\pi(F)(x) = f_1(x) f_2(x)$$

where $f_1(x), f_2(x) \in k[x]$ are relatively prime monic polynomials.
Question: Are there monic polynomials $F_1(x), F_2(x) \in A[x]$ such that

$$F(x) = F_1(x) F_2(x) \quad \text{and} \quad f_1 = \pi(F_1), f_2 = \pi(F_2)?$$

If this question is answered in the affirmative for all $F(x) \in A[x]$, the local ring (A, m) is called *Henselian*.

Henselian rings – examples

- The local ring $A = k[t]_t$, where k is a field with $\mathrm{char}(k) > 2$, is not Henselian. For example, the polynomial

$$F(x) = x^2 - t - 1$$

  has a factorization in $k[x]$ but not in $A[x]$.
- The local ring $k[[t]]$, where k is a field, is Henselian.
- The ring of p-adic integers is Henselian.
- In general, one can show that a (commutative) local ring (A, m) which is Hausdorff and complete in the m-adic topology is Henselian.

A non-commutative analogue of the concept of Henselian rings

Let (A, m) be a (not necessarily commutative) local ring, $k = A/m$ be the residue (skew) field of A and $\pi : A \to k$ be the quotient map. We assume that k is commutative. Let $A[x]$ be the ring of polynomials over A where x commutes with all elements of A. Suppose that $F(x) \in A[x]$ is a monic polynomial for which we have a factorization

$$\pi(F)(x) = f_1(x) f_2(x)$$

where $f_1(x), f_2(x) \in k[x]$ are relatively prime monic polynomials.
Question: Are there monic polynomials $F_1(x), F_2(x) \in A[x]$ such that

$$F(x) = F_1(x) F_2(x) \text{ and } f_1 = \pi(F_1), f_2 = \pi(F_2)?$$

If this question is answered in the affirmative for all $F(x) \in A[x]$, the local ring (A, m) is called *Henselian*.

Non-commutative Henselian rings – examples

- Let k be a commutative field with a nontrivial derivation, e.g. $\mathbb{C}(x)$ with $\frac{d}{dx}$. The ring of Volterra operators $k[[\partial^{-1}]]$ is the set of formal series

$$a_0 + a_1\partial^{-1} + \cdots + a_n\partial^{-n} + \cdots$$

  where $a_0, a_1, \ldots \in k$, and $\partial^{-1}a = \sum_{n=0}^{\infty}(-1)^n a^{(n)}\partial^{-1-n}$. The ring $k[[\partial^{-1}]]$ is a non-commutative Henselian ring.

- Consider the ring T of "twisted" power series on $\mathbb{C}$, i.e.

$$c_0 + c_1\tau + \cdots + c_n\tau^n + \cdots$$

  where $c_0, c_1, \ldots \in \mathbb{C}$, and $\tau c = \bar{c}\tau$ for every $c \in \mathbb{C}$. The ring T is a non-commutative local ring whose residue field is commutative. The ring T is not Henselian.
  Remark Both $k[[\partial^{-1}]]$ and T are Hausdorff and complete in their m-adic topology.

A theorem regarding non-commutative Henselian rings

Let (A, m) be a (not necessarily commutative) local ring. The filtration

$$\cdots \subset m^{n+1} \subset m^n \subset \cdots \subset m \subset A$$

gives rise to the associated graded ring

$$gr_m(A) = \frac{A}{m} \oplus \frac{m}{m^2} \oplus \cdots \oplus \frac{m^n}{m^{n+1}} \oplus \cdots$$

where $(a + m^{i+1})(b + m^{j+1}) := ab + m^{i+j+1}$. The ring (A, m) is called *almost commutative* if $gr_m(A)$ is commutative.

Theorem

Let (A, m) be a (not necessarily commutative) local ring. Suppose that A is Hausdorff and complete in its m-adic topology. If A is almost commutative, then A is Henselian.

A characterization of non-commutative Henselian rings

It is known that a commutative local ring A is Henselian if and only if every finite A-algebra is isomorphic to a product of local rings.
In the non-commutative case, we have the following characterization

Theorem

Let A be an almost commutative local ring. Then, A is Henselian if and only if for every monic polynomial $p \in A[x]$, the $A[x]$-module $\frac{A[x]}{A[x]p}$ is a finite direct sum of local A-modules.

Henselization

"It is well known that, for any commutative local Noetherian ring A, there is a commutative Henselian ring A_h and a local homomorphism $i : A \to A_h$ with the following universal property: given any local homomorphism f from A to some [commutative] Henselian ring B there is a unique local homomorphism $f_h : A_h \to B$ such that $f = f_h i$."[1]
The pair (A_h, i) is unique up to isomorphism and is called the *Henselization* of A.

Open problem: Does evey almost commutative Notherian local ring have a Henselization?

[1] Aryapoor, M., Non-commutative Henselian rings, J. Algebra 322 (2009), no. 6

Thank you!