Prime nearly epsilon-strongly graded rings

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Prime rings

Definition

A non-commutative, non-unital ring is called *prime* if (0) is a prime ideal.

Example

- Commutative ring is prime if and only if ID;
- $R \times R$ is not prime;
- $M_n(R)$ is prime if R is prime.

The group ring

Definition

Let R be a unital ring and let G be a group. The *group ring* $R[G] := \bigoplus_{g \in G} R\delta_g$ is a unital ring with multiplication defined by linearly extending

$$\delta_g \delta_h = \delta_{gh}$$

for all $g, h \in G$.

Connell's Theorem (1963)

The group ring $R[G]$ is prime if and only if R is prime and G does not have any non-trivial finite normal subgroups.

Group graded rings (I)

Definition

Let G be a group and let S be a ring. A *grading* of S is a collection of additive subsets of S, $\{S_g\}_{g\in G}$, such that

$$S = \bigoplus_{g\in G} S_g,$$

and $S_g S_h \subseteq S_{gh}$ for all $g, h \in G$. The ring S is called a *G-graded ring*. The component S_e is a subring of S called the *principal component*.

Definition (Dade 1980, Năstăsescu-Oystaeyen, 1982)

A G-graded ring is called *strongly* G-graded if $S_g S_h = S_{gh}$ for all $g, h \in G$.

Group graded rings (II)

Example

The group ring $R[G]$ is strongly G-graded with $S_g := R\delta_g$.

Example

The R be a unital ring. An *algebraic crossed product* $R \star G := \bigoplus_{g \in G} R\delta_g$ generalizes the group ring by slightly modifying the multiplication.

Example (Hazrat, Nystedt-Öinert)

The Leavitt path algebra $L_K(E)$ is $\mathbb{Z}$-graded but *not* strongly $\mathbb{Z}$-graded in general!

Prime strongly graded rings (I)

Let $S = \bigoplus_{g \in G} S_g$ be a unital strongly G-graded ring.

Definition (Passman, 1984)

G acts on the ideals of S_e by

$$I^g := S_{g^{-1}} I S_g.$$

An ideal I of S_e is called *G-invariant* if $I^g = I$ for every $g \in G$.

Prime strongly graded rings (II)

Theorem (Passman 1984)

If S is a unital and strongly G-graded ring, then S is not prime if and only if there exist:

(a) *subgroups $N \lhd H \subseteq G$ with N finite,*

(b) *an H-invariant ideal I of S_e with $I^g I = \{0\}$ for all $g \in G \setminus H$,*

(c) *nonzero H-invariant ideals $\tilde{A}$ and $\tilde{B}$ of S_N with $\tilde{A}, \tilde{B} \subseteq IS_N$ and $\tilde{A}\tilde{B} = \{0\}$.*

Remark

- Passman's characterization generalizes Connell's Theorem.
- We want: further generalization to nearly epsilon-strongly graded rings!

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Nearly epsilon-strongly graded rings

Definition (Nystedt-Öinert, 2017)

Let S be a G-graded ring. Suppose that $S_g S_{g^{-1}}$ is an s-unital ideal of S_e for every $g \in G$, and,

$$S_g = S_g S_{g^{-1}} S_g \tag{1}$$

for every $g \in G$. Then S is called *nearly epsilon-strongly G-graded*.

Proposition

If S is an s-unital strongly G-graded, then S is nearly epsilon-strongly G-graded.

Proof.

Note that $S_g S_{g^{-1}} = S_e$ for every $g \in G$ by the strongly graded property. □

Nearly epsilon-strongly graded rings (II)

Example

- *unital* strongly G-graded rings. E.g. the group ring $R[G]$.
- Leavitt path algebras (Nystedt-Öinert, 2017).
- Algebraic *partial* crossed products (Nystedt-Öinert-Pinedo, 2016)
- Corner skew Laurent polynomial rings (Lännström, 2019).

Our partial characterization

Theorem/WIP (Lännström, Nystedt, Öinert, Wagner, 2020)

Let S be a nearly epsilon-strongly G-graded ring. Suppose that either of these conditions are satisfied:

- $r.\text{Ann}_S(S_x) = 0$ for every $x \in \text{Supp}(S)$, or,
- G is an FC-group.

Then we have obtained sufficient and necessary conditions (a)-(c) for S to be prime (not included here).

Remark

- Conditions (a)-(c) are similar to Passman's.
- If S is unital strongly G-graded, then $r.\text{Ann}_S(S_x) = 0$ for every $x \in \text{Supp}(S)$.

Applications of our characterization

Remark

By applying our characterizing, we can determine when:

- $R[G]$ is prime (i.e. Connell's Theorem), and when,
- Leavitt path algebras are prime (Abrams, Bell and Rangaswamy).

Corollary (Lännström, Nystedt, Öinert, Wagner, 2020)

The unital partial crossed product $R \star G$ is prime if R is prime and G is torsion-free.

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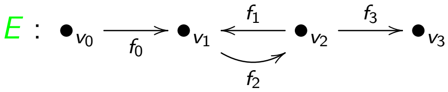
Leavitt path algebras

Introduced by Ara, Moreno and Pardo 2004 and by Abrams and Aranda Pino 2005.

Input to the construction:

1. R be a unital ring (possibly non-commutative)
2. E be a directed graph. E.g.:



The Leavitt path algebra $L_R(E)$ is a $\mathbb{Z}$-graded R-algebra.

Research questions/motif

Coefficients in a field (original construction), coefficients in a commutative unital ring (Tomforde, 2009), coefficients in a unital ring (Hazrat, 2013)

Research questions 1

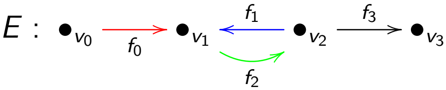
How does algebraic properties of R effect the algebraic properties of $L_R(E)$?

Research question 2

Can we extend structural results about $L_K(E)$ to $L_R(E)$ where R is a general unital ring?

Leavitt path algebras: Examples I

Ex: Consider the LPA associated with



Elements in $L_R(E)$:
$\alpha^* = f_1^* f_2^* f_0^* \in L_R(E)$
$v_0 \in L_R(E)$
$\gamma = f_0 \in L_R(E)$

$$\alpha^* \gamma = f_1^* f_2^* f_0^* f_0 = f_1^* f_2^* r(f_0) = f_1^* f_2^*.$$

Leavitt path algebras: Examples II

Example

$$A_1 : \qquad \bullet_v$$

In this case, $L_R(A_1) \cong Rv \cong R$.

Example

E_1 :

In this case, $L_R(E_1) \cong_\phi R[x, x^{-1}]$ via the map defined by $\phi(v) = 1_R, \phi(f) = x, \phi(f^*) = x^{-1}$.

Leavitt path algebras: Examples III

The previous graphs have all been finite, but we also allow infinite graphs!

Example

Infinitely many vertices:

$$E' : \bullet_{v_1} \quad \bullet_{v_2} \quad \bullet_{v_3} \quad \bullet_{v_4} \quad \bullet_{v_5} \quad \bullet_{v_6} \quad \bullet_{v_7} \quad \bullet_{v_8} \quad \bullet_{v_9} \quad \bullet_{v_{10}} \cdots$$

In this case, $L_R(E') \cong \bigoplus_{i>0} R v_i$.

Example

$$E'' : \bullet_{v_1} \xrightarrow{(\infty)} \bullet_{v_2}$$

Prime Leavitt path algebras

Definition

A directed graph E satisfied Condition (MT-3) if for every pair u, v of vertices, there is some vertex w such that there are paths:

1. $u \to w$, and,
2. $v \to w$.

Theorem (Abrams, Bell and Rangaswamy)

Let K be a field. The Leavitt path $L_K(E)$ is prime if and only if E satisfies Condition (MT-3).

Examples

Example

Let R be a unital ring.

$$A_2 : \quad \bullet_{v_1} \qquad \bullet_{v_2}$$

In this case, $L_R(A_2) \cong Rv_1 \oplus Rv_2 \cong R \times R$.

Remark

- A_2 does not satisfy (MT-3)
- Remember that $R \times R$ is never prime!

Examples

Example

Let R be a unital ring.

$$E_2 : \quad \bullet_{v_1} \longrightarrow \bullet_{v_2}$$

In this case, $L_R(E_2) \cong M_2(R)$.

Remark

- E_2 does satisfy (MT-3)
- Remember that $M_2(R)$ is prime if R is prime

Our extended characterization of prime Leavitt path algebras

Theorem (Lännström, Nystedt, Öinert, Wagner, 2020)

Let R be a unital ring. The Leavitt path $L_R(E)$ is prime if and only if R is prime and E satisfies Condition (MT-3).

Remark

Note that this extends the Abrams- Bell-Rangaswamy theorem to also include Leavitt path algebras over unital rings!

Thank you for your attention!