On Solvability and Nilpotency of n-Hom-Lie algebras

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Outline

- Introduction and context.
- Basic definitions and properties of n-Hom-Lie algebras.
- Derived series and central descending series of n-Hom-Lie algebras and their properties.
- Relation to algebra twisting.

This talk is based on a work done in collaboration with Sergei Silvestrov and Abdenacer Makhlouf.

Introduction and context

- Filippov V. T., n-Lie algebras (1985)
- Kasymov Sh. M., Theory of n-Lie algebras (1987)
- Hartwig J. T., Larsson D., Silvestrov S. D., Deformations of Lie algebras using σ-derivations (2003)
- Larsson D., Silvestrov S., Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities (2005)
- Makhlouf A., Silvestrov S. D., Hom-algebra structures (2006)
- Ataguema H., Makhlouf A., Silvestrov S., Generalization of n-ary Nambu algebras and beyond (2009)
- Yau D., On n-ary Hom-Nambu and Hom-Nambu-Lie algebras (2012)

All vector spaces are considered over a field of characteristic 0.

Definition

An n-Hom-Lie algebra is a vector space A together with a skew-symmetric n-linear map $[\cdot, ..., \cdot]$ and $(n-1)$ linear maps $\alpha_i, 1 \leq i \leq n-1$ defined on A satisfying the Hom-Nambu-Filippov identity:

$$[\alpha_1(x_1), ..., \alpha_{n-1}(x_{n-1}), [y_1, ..., y_n]] =$$
$$\sum_{i=1}^{n} [\alpha_1(y_1), ..., \alpha_{i-1}(y_{i-1}), [x_1, ..., x_{n-1}, y_i], \alpha_i(y_{i+1}), ..., \alpha_{n-1}(y_n)],$$
$$\forall x_1, ..., x_{n-1}, y_1, ..., y_n \in A.$$

Definition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$, $(B, \{\cdot, ..., \cdot\}, \beta_1, ..., \beta_{n-1})$ be n-Hom-Lie algebras. An n-Hom-Lie algebra morphism is a linear map $f : A \to B$ satisfying the conditions:

- $f([x_1, ..., x_n]) = \{f(x_1), ..., f(x_n)\}$, for all $x_1, ..., x_n \in A$.
- $f \circ \alpha_i = \beta_i \circ f$, for all $i : 1 \leq i \leq n-1$.

A linear map satisfying only the first condition is called a weak morphism.

Definition

We refer to an n-Hom-Lie algebra $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ such that $\alpha_1 = \alpha_2 = ... = \alpha_{n-1} = \alpha$ by $(A, [\cdot, ..., \cdot], \alpha)$.

- It is said to be multiplicative if α is an algebra morphism.
- It is said to be regular if it is multiplicative and α is an isomorphism.

Definition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ be an n-Hom-Lie algebra. Let B be a subspace of A invariant under all the linear maps α_i:

- If for all $x_1, ..., x_n \in B$ we have $[x_1, ..., x_n] \in B$, then B is a subalgebra of A.
- If for all $x_1, ..., x_{n-1} \in A$, and $y \in B$ we have $[x_1, ..., x_{n-1}, y] \in B$, then B is an ideal of A.

If we drop the invariance under the twisting maps, B will be called a weak subalgebra or a weak ideal respectively.
A good way to see the difference is to consider factor algebras with respect to ideals and weak ideals.

The following result presents a way to construct n-Hom-Lie algebras from n-Lie algebras by "twisting":

Proposition

Let $(A, [\cdot, ..., \cdot], \alpha)$ be an n-Hom-Lie algebra, $\beta : A \to A$ an algebra weak morphism, we define $[\cdot, ..., \cdot]_\beta$ by:

$$[x_1, ..., x_n]_\beta = \beta([x_1, ..., x_n]).$$

We have that $\left(A, [\cdot, ..., \cdot]_\beta, \beta \circ \alpha\right)$ is an n-Hom-Lie algebra. Moreover if $(A, [\cdot, ..., \cdot], \alpha)$ is multiplicative and $\beta \circ \alpha = \alpha \circ \beta$ then $\left(A, [\cdot, ..., \cdot]_\beta, \beta \circ \alpha\right)$ is multiplicative.

A particular case of interest would be when $\alpha = Id_A$, that is A is an n-Lie algebra.

Now, we present the various derived series and central descending series for n-Hom-Lie algebras, and their basic properties:

Definition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ be an n-Hom-Lie algebra, and let I be an ideal of A.For $2 \leq k \leq n$, we define the k-derived series of the ideal I by:

$$D_k^0(I) = I \text{ and } D_k^{r+1} = \left[\underbrace{D_k^p(I), ..., D_k^p(I)}_{k}, \underbrace{A, ..., A}_{n-k} \right].$$

We define the k-central descending series of I by:

$$C_k^0(I) = I \text{ and } C_k^{p+1}(I) = \left[C_k^p(I), \underbrace{I, ..., I}_{k-1}, \underbrace{A, ..., A}_{n-k} \right].$$

We look at how these definitions look like for $k = n$, $k = 2$ and then for $n = 2$:

$$D_n^0(I) = I \text{ and } D_n^{r+1} = \left[D_n^p(I), ..., D_n^p(I)\right].$$

$$C_n^0(I) = I \text{ and } C_n^{p+1}(I) = \left[C_n^p(I), I, ..., I\right].$$

$$D_2^0(I) = I \text{ and } D_2^{r+1} = \left[D_2^p(I), D_2^p(I), A, ..., A\right].$$

$$C_2^0(I) = I \text{ and } C_2^{p+1}(I) = \left[C_2^p(I), I, A, ..., A\right].$$

$$D^0(I) = I \text{ and } D^{r+1} = \left[D^p(I), D^p(I)\right].$$

$$C^0(I) = I \text{ and } C^{p+1}(I) = \left[C^p(I), I\right].$$

Lemma

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ *be an* n*-Hom-Lie algebra, and let* I *be an ideal of* A*. For* $2 \leq k \leq n-1$ *and* $r \in \mathbb{N}$*, we have that*

$$D^r_{k+1}(I) \subseteq D^r_k(I) \text{ and } C^r_{k+1}(I) \subseteq C^r_k(I).$$

We also have:

$$C^{r+1}_k(I) \subseteq C^r_k(I) \text{ and } D^{r+1}_k(I) \subseteq D^r_k(I).$$

Proposition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ *be an* n*-Hom-Lie algebra, and let* I *be an ideal of* A*. For all* $2 \leq k \leq n$ *and all* $r \in \mathbb{N}$*, we have that* $D^r_k(I)$ *and* $C^r_k(I)$ *are weak subalgebras of* A*.*

Proposition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ be an n-Hom-Lie algebra, and let I be an ideal of A. For all $2 \leq k \leq n$ and all $r \in \mathbb{N}$. If all the linear maps $\alpha_i, 1 \leq i \leq n-1$ are weak morphisms, then the subspaces $D_k^r(I)$ and $C_k^r(I)$ are subalgebras of A. If, in addition, these maps are surjective, then $D_k^r(I)$ and $C_k^r(I)$ are ideals of A.

Definition

Let $(A, [\cdot, ..., \cdot], \alpha_1, ..., \alpha_{n-1})$ be an n-Hom-Lie algebra, and let I be an ideal of A. For $2 \leq k \leq n$, the ideal I is said to be k-solvable (resp. k-nilpotent) if there exists $r \in \mathbb{N}$ such that $D_k^r(I) = \{0\}$ (resp. $C_k^r(I) = \{0\}$). in this case, the smallest $r \in \mathbb{N}$ satisfying this condition is called the class of k-solvability (resp. the class of nilpotence) of I.

Corollary

If an ideal I of an n-Hom-Lie algebra is k-solvable (resp. k-nilpotent) then it is l-solvable (resp. l-nilpotent) for all $l > k$.

The following result shows the preservation of the derived series and central descending series under isomorphisms.

Proposition

Let $(B, [\cdot, ..., \cdot], \beta_1, ..., \beta_{n-1})$ another n-hom-Lie algebra, $f : A \to B$ a surjective algebra morphism and I an ideal of A. Then for all $r \in \mathbb{N}$, for all $2 \leq k \leq n$, we have:

$$f\left(D_k^r(I)\right) = D_k^r\left(f(I)\right) \text{ and } f\left(C_k^r(I)\right) = C_k^r\left(f(I)\right).$$

Now we proceed to show that the k-solvability is a radical property:

Proposition

Let I, J be ideals of A such that $J \subseteq I$. If J is k-solvable and $\bar{I}$ defined by

$$\bar{I} = \{\bar{i} = i + J, i \in I\}$$

is k-solvable in $\frac{A}{J}$ then I is k-solvable.

Corollary

Let I_1, I_2 be two k-solvable ideals of A. Then $I_1 + I_2$ is a k-solvable ideal of A.

Based on this, we can define the following:

Definition

Suppose that A is finite dimensional. The greatest k-solvable ideal of A, $Rad_k(A)$, that is the sum of all k-solvable ideals of A is called the k-radical of A. If $Rad_k(A)$ is trivial, A is said to be k-semisimple.

Proposition

Let A be a finite dimensional n-Hom-Lie algebra. The factor algebra $\frac{A}{Rad_k(A)}$ is k-semisimple.

Remark

We have, for all $2 \leq k \leq k' \leq n$, $Rad_k(A) \subseteq Rad_{k'}(A)$, which implies that if A is (k')-semisimple then it is k-semisimple. This follows from the relation between k-solvability and k'-solvability.

Relation to algebra twisting

We now look at the particular case of n-Hom-Lie algebras obtained by twisting, let $A = (A, [\cdot, ..., \cdot], \beta$ be an n-Hom-Lie algebra, α a weak endomorphism and $A_\alpha = (A, [\cdot, ..., \cdot]_\alpha, \alpha \circ \beta)$ the resulting n-Hom-Lie algebra. Then we have:

Lemma

Let I be an ideal of A. If I is invariant under α then it is an ideal of A_α.

Lemma

Let I be an ideal of A invariant under α. For all $r \in \mathbb{N}$, $2 \leq k \leq n$,

$$\alpha\left(D_k^r(I)\right) \subseteq D_k^r(I) \text{ and } \alpha\left(C_k^r(I)\right) \subseteq C_k^r(I).$$

Relation to algebra twisting

Proposition

Let I be an ideal of A invariant under α. For all $r \in \mathbb{N}$, $2 \leq k \leq n$, we have:

$$D_k^r(I)_\alpha \subseteq D_k^r(I) \text{ and } C_k^r(I)_\alpha \subseteq C_k^r(I),$$

where $D_k^r(I)_\alpha$ and $C_k^r(I)_\alpha$ represent the terms of the k-derived series and k-central descending series in A_α.

Relation to algebra twisting

Remark

If α is invertible, by applying the proposition above for α and α^{-1} we get:

$$D_k^r(I)_\alpha = D_k^r(I) \text{ and } C_k^r(I)_\alpha = C_k^r(I),$$

Corollary

Let I be an ideal of A invariant under α. If I is k-solvable (resp. k-nilpotent) in A_α then it is k-solvable (resp. k-nilpotent) in A. Moreover, if α is bijective, then I is k-solvable (resp. k-nilpotent) in A_α if and only if it is k-solvable (resp. k-nilpotent) in A.

The End

Thank you