Twisted connections on projective modules

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Introduction

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for $f, g \in \mathcal{A}$, and introduced a (σ, τ)-connection ∇, fulfilling a corresponding twisted Leibniz rule

$$\nabla_X(fm) = \sigma(f)\nabla_X(m) + X(f)\hat{\tau}(m)$$

for $f \in \mathcal{A}$ and elements m in a (left) $\mathcal{A}$-module M, where $\hat{\tau}$ is an extension of τ to M (to be defined later).

Introduction

Moreover, we introduced corresponding concepts of metric compatibility and torsion-freeness of such a connection.

We proved that there exists a class of metric and torsion-free connections ("Levi-Civita connections") on the (standard) module of differential forms over S_q^3.

In this talk I will report on current work on extending these ideas to general algebras and (σ, τ)-derivations.

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    $$X(fg) = \sigma(f)X(g) + X(f)\tau(g)$$
    for $f, g \in \mathcal{A}$.
- A (σ, τ)-derivation can be regarded as a twisted derivation from which we can construct a twisted connection.

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  - constructing a connection $\nabla_{X_a} : M \to M$ satisfying a Leibniz rule

    $$\nabla_{X_a}(fm) = \sigma_a(f)\nabla_{X_a}(m) + X_a(f)\hat{\tau}_a(m)$$

    for $a \in I$, $f \in \mathcal{A}$ and $m \in M$ where $\sigma_a, \tau_a, : \mathcal{A} \to \mathcal{A}$ are algebra endomorphisms and $\hat{\tau}_a : M \to M$ is a map such that

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- Finally, we would like to call such a connection a (σ, τ)-connection and show that it exists on projective modules.

(σ, τ)-algebra

Definition 1

Let $\mathcal{A}$ be an associative algebra and let σ and τ be endomorphisms of $\mathcal{A}$. A $\mathbb{C}$ - linear map $X : \mathcal{A} \to \mathcal{A}$ is called a (σ, τ)-derivation if

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Definition 2

A (σ, τ)-*algebra* $\Sigma = (\mathcal{A}, \{X_a\}_{a \in I})$ is a pair where $\mathcal{A}$ is an associative algebra (over $\mathbb{C}$) and X_a is a (σ_a, τ_a)-derivation of $\mathcal{A}$ for $a \in I$.

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Definition 3

For a (σ, τ)-algebra $\Sigma = (\mathcal{A}, \{X_a\}_{a \in I})$ we let

$$T\Sigma \subseteq \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{A})$$

be the vector space generated by $\{X_a\}_{a \in I}$. We call $T\Sigma$ the *tangent space of Σ*.

Σ-module

Definition 4

Let $\Sigma = (\mathcal{A}, \{X_a\}_{a \in I})$ be a (σ, τ)-algebra. A *left Σ-module* $(M, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a \in I})$ is a left $\mathcal{A}$-module M together with $\mathbb{C}$-linear maps $\hat{\sigma}_a, \hat{\tau}_a : M \to M$ such that

$$\hat{\sigma}_a(fm) = \sigma_a(f)\hat{\sigma}_a(m)$$
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for $f \in \mathcal{A}$, $m \in M$ and $a \in I$.

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Definition 5

Let $\Sigma = (\mathcal{A}, \{X_a\}_{a\in I})$ be a (σ, τ)-algebra and let $(M_1, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a\in I})$ and $(M_2, \{(\tilde{\sigma}_a, \tilde{\tau}_a)\}_{a\in I})$ be left Σ-modules. A (σ, τ)*-module homomorphism* is an $\mathcal{A}$-module homomorphism $\phi : M_1 \to M_2$ such that

$$\phi(\hat{\sigma}_a(m)) = \tilde{\sigma}_a(\phi(m)) \quad \phi(\hat{\tau}_a(m)) = \tilde{\tau}_a(\phi(m))$$

for $m \in M_1$ and $a \in I$.

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Let $\Sigma = (\mathcal{A}, \{X_a\}_{a \in I})$ be a (σ, τ)-algebra and let $\mathcal{A}^n$ be a free (left) $\mathcal{A}$-module with a basis $e_1, \ldots, e_n$.

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One can introduce a canonical Σ-module structure on $\mathcal{A}^n$ by setting

$$\hat{\sigma}_a^0(m) = \sigma_a(m^i)e_i, \quad \hat{\tau}_a^0(m) = \tau_a(m^i)e_i$$

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One has

$$\hat{\sigma}^0_a(fm) = \sigma_a(fm^i)e_i = \sigma_a(f)\sigma_a(m^i)e_i = \sigma_a(f)\hat{\sigma}^0_a(m)$$
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showing that $(\mathcal{A}^n, \{(\hat{\sigma}^0_a, \hat{\tau}^0_a)\}_{a\in I})$ is a (left) Σ-module.

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showing that $(p\mathcal{A}^n, \{(\tilde{\sigma}_a, \tilde{\tau}_a)\}_{a\in I})$ is a Σ-module. Hence, every projective $\mathcal{A}$-module can be endowed with the structure of a Σ-module.

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$$(p\mathcal{A}^n, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a\in I}) \simeq (M, \{(\tilde{\sigma}_a, \tilde{\tau}_a)\}_{a\in I})$$

and furthermore, $[\hat{\sigma}_a, p] = [\hat{\tau}_a, p] = 0.$

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$$p^2 = \psi \circ \phi \circ \psi \circ \phi = \psi \circ \phi = p,$$

since $\phi \circ \psi = \mathrm{Id}_M$. This shows that p is a projection and $p\mathcal{A}^n$ is a projective module.

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since $\phi \circ \psi = \mathrm{Id}_M$. This shows that p is a projection and $p\mathcal{A}^n$ is a projective module. Let $\hat{\phi} = \phi|_{p\mathcal{A}^n} : p\mathcal{A}^n \to M$ be the restriction of ϕ to $p\mathcal{A}^n$. One has

$$p(\psi(m)) = \psi \circ \phi \circ \psi(m) = \psi(m),$$

showing that $\psi(m) \in p\mathcal{A}^n$ and $\hat{\phi}(\psi(m)) = m$.

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$$\begin{aligned}\hat{\sigma}_a(fm) &= \psi(\tilde{\sigma}_a(\hat{\phi}(fm))) = \psi(\tilde{\sigma}_a(f\hat{\phi}(m))) = \psi(\sigma_a(f)\tilde{\sigma}_a(\hat{\phi}(m))) \\ &= \sigma_a(f)\psi(\tilde{\sigma}_a(\hat{\phi}(m))) = \sigma_a(f)\hat{\sigma}_a(m).\end{aligned}$$

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$$= \sigma_a(f)\psi(\tilde{\sigma}_a(\hat{\phi}(m))) = \sigma_a(f)\hat{\sigma}_a(m).$$

Similarly,

$$\hat{\tau}_a(fm) = \psi(\tilde{\tau}_a(\hat{\phi}(fm))) = \psi(\tilde{\tau}_a(f\hat{\phi}(m))) = \psi(\tau_a(f)\tilde{\tau}_a(\hat{\phi}(m)))$$
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This shows that $(p\mathcal{A}^n, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a\in I})$ is a Σ-module.

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$$= \tau_a(f)\psi(\tilde{\tau}_a(\hat{\phi}(m))) = \tau_a(f)\hat{\tau}_a(m).$$

This shows that $(p\mathcal{A}^n, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a\in I})$ is a Σ-module. In fact one has

$$\hat{\phi} \circ \hat{\sigma}_a = \hat{\phi} \circ \psi \circ \tilde{\sigma}_a \circ \hat{\phi} = \tilde{\sigma}_a \circ \hat{\phi}$$
$$\hat{\phi} \circ \hat{\tau}_a = \hat{\phi} \circ \psi \circ \tilde{\tau}_a \circ \hat{\phi} = \tilde{\tau}_a \circ \hat{\phi},$$

showing that $\hat{\phi}$ is a (σ, τ)- isomorphism. □

Proof.

Let $\psi(m) \in p\mathcal{A}^n$. Using $\hat{\phi} \circ \psi = \text{id}$, one computes

$$\hat{\sigma}_a \circ p = \psi \circ \tilde{\sigma}_a \circ \hat{\phi} \circ \psi \circ \hat{\phi} = \psi \circ \tilde{\sigma}_a \circ \hat{\phi}.$$

and

$$p \circ \hat{\sigma}_a = \psi \circ \hat{\phi} \circ \psi \circ \tilde{\sigma}_a \circ \hat{\phi} = \psi \circ \tilde{\sigma}_a \circ \psi$$

giving $[\hat{\sigma}_a, p] = 0$. In the similar way one can show that $[\hat{\tau}_a, p] = 0$. □

(σ, τ)-connection

Definition 10

Let $\Sigma = (\mathcal{A}, \{X_a\}_{a \in I})$ be a (σ, τ)-algebra and let $(M, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a \in I})$ be a left Σ-module. A left *(σ, τ)-connection on M* is a map $\nabla : T\Sigma \times M \to M$ satisfying

$$
\begin{aligned}
&\nabla_X(m + m') = \nabla_X m + \nabla_X m' \\
&\nabla_X(\lambda m) = \lambda \nabla_X m \\
&\nabla_{X+Y} m = \nabla_X m + \nabla_Y m \\
&\nabla_{\lambda X} m = \lambda \nabla_X m \\
&\nabla_{X_a}(fm) = \sigma_a(f) \nabla_{X_a} m + X_a(f) \hat{\tau}_a(m)
\end{aligned}
$$

for all $X, Y \in T\Sigma$, $m, m' \in M$, $\lambda \in \mathbb{C}$ and $a \in I$.

Example 11

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$$\nabla_X(m^i e_i) = \sigma_a(m^i)\nabla_X e_i + X(m^i)\hat{\tau}_a(e_i).$$

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For every $X, Y \in T\Sigma$, one can easily see that

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For derivations $X_a \in T\Sigma$, one finds that

$$\nabla_{X_a}(fm) = \sigma_a(f)\nabla_{X_a}(m) + X_a(f)\hat{\tau}_a(m)$$

for $m, m' \in \mathcal{A}^n$ and $f \in \mathcal{A}$.

Proposition 12

Let $(M, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a\in I})$ *be a left* Σ*-module and let* ∇ *be a left* (σ, τ)*-connection on* $(M, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a\in I})$*. If* $p : M \to M$ *is a projection then* $\tilde{\nabla} = p \circ \nabla$ *is a left* (σ, τ)*-connection on* $(p(M), \{(p \circ \hat{\sigma}_a, p \circ \hat{\tau}_a)\}_{a\in I})$*.*

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$$\begin{aligned}
\tilde{\nabla}_{X_a}(fm) &= p(\nabla_{X_a}(fm)) \\
&= p(\sigma_a(f)\nabla_{X_a} m) + p(X_a(f)\hat{\tau}_a(m)) \\
&= \sigma_a(f)p(\nabla_{X_a} m) + X_a(f)p(\hat{\tau}_a(m)) \\
&= \sigma_a(f)\tilde{\nabla}_{X_a} m + X_a(f)p\circ\hat{\tau}_a(m).
\end{aligned}$$

□

To show linearity of the connection on the projective module we have

$$\begin{aligned}\tilde{\nabla}_{X_a}(\lambda m + m') &= p(\lambda \nabla_{X_a}(m)) + p(\nabla_{X_a}(m')) \\ &= \lambda \tilde{\nabla}_{X_a}(m) + \tilde{\nabla}_{X_a}(m')\end{aligned}$$

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Let $(M, \{(\tilde{\sigma}_a, \tilde{\tau})\}_{a \in I})$ be a Σ-module with M be a projective module. By proposition (9), there exist a projection $p : \mathcal{A} \to \mathcal{A}$ such that

$$(p\mathcal{A}^n, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a \in I}) \simeq (M, \{(\tilde{\sigma}_a, \tilde{\tau}_a)\}_{a \in I}).$$

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Let ∇ be a (σ, τ)-connection on $(\mathcal{A}^n, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a \in I})$ and define $\tilde{\nabla} = p \circ \nabla$.

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Let ∇ be a (σ, τ)-connection on $(\mathcal{A}^n, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a \in I})$ and define $\tilde{\nabla} = p \circ \nabla$. By proposition (12), $\tilde{\nabla}$ is a (σ, τ)-connection on $(p\mathcal{A}^n, \{(\hat{\sigma}_a, \hat{\tau}_a)\}_{a \in I})$. By the isomorphism, it is clear that the projective Σ-module $(M, \{(\tilde{\sigma}_a, \tilde{\tau})\}_{a \in I})$ has a (σ, τ)-connection. $\square$

outlook

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- on $(\sigma, \tau)-$metric connection on Σ-bimodule.
- the general case of torsion and curvature since we have shown in [AIL20] that a Levi-Civita connection exists on S_q^3.

Thank you very much for your attention.