The hom-associative Weyl algebras in prime characteristic

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Introduction

Many Lie algebras are *rigid*; they cannot be deformed without altering the Jacobi identity (e.g. any semisimple Lie algebra in characteristic zero is rigid). Remedy: generalize Lie algebras into *hom-Lie algebras*, as introduced in [HLS06]. In this context, *hom-associative algebras* arise naturally.

Another remedy: deform the universal enveloping algebra $U(L)$ of the Lie algebra L. But $U(L)$ can also be rigid as an associative algebra (L is *strongly rigid*). However, $U(L)$ need not be rigid as a hom-associative algebra!

[HLS06] J.T. Hartwig, D. Larsson, and S.D. Silvestrov. "Deformations of Lie algebras using σ-derivations". In: *J. Algebra* 295.2 (2006).

Non-commutative polynomial rings – or *Ore extensions* – were introduced by Ore [Ore33], and recently generalized to the non-associative setting [NÖR18] and the hom-associative setting [BRS18], independently.

Ore extensions include many rigid algebras, e.g. rigid universal enveloping algebras of Lie algebras, and the Weyl algebras in characteristic zero. These can now be deformed, as hom-associative Ore extensions.

This talk is about a nasty version of the hom-associative Weyl algebras – the hom-associative Weyl algebras in *prime characteristic* [BR20b].

[Ore33] O. Ore. "Theory of Non-Commutative Polynomials". In: *Ann. Math.* 34.3 (1933).

[NÖR18] P. Nystedt, J. Öinert, and J. Richter. "Non-associative Ore extensions". In: *Isr. J. Math.* 224.1 (2018).

[BRS18] P. Bäck, J. Richter, and S. Silvestrov. "Hom-associative Ore extensions and weak unitalizations". In: *Int. Electron. J. Algebra* 24 (2018).

[BR20b] P. Bäck and J. Richter. "The hom-associative Weyl algebras in prime characteristic". Working paper. 2020.

Hom-algebras

Definition (Hom-everything)

A *hom-associative algebra* over an associative, commutative, and unital ring R, is a triple $(M, \cdot, \alpha)$ consisting of an R-module M, an R-bilinear map $\cdot\colon M \times M \to M$, and an R-linear map $\alpha\colon M \to M$, satisfying,

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c), \quad \forall a, b, c \in M.$$

A *hom-associative ring* is a hom-associative algebra over $\mathbb{Z}$.

A map $f\colon A \to B$ between hom-associative algebras is a *homomorphism* if it is linear, multiplicative, and $f \circ \alpha_A = \alpha_B \circ f$.

A left (right) ideal I s.t. $\alpha(I) \subseteq I$ is a left (right) *hom-ideal*.

Definition (Weakly unital hom-associative algebra)

A hom-associative algebra A is called *weakly unital* with *weak unit* $e \in A$ if for all $a \in A$, $e \cdot a = a \cdot e = \alpha(a)$.

Proposition ([BRS18])

Any multiplicative hom-associative R-algebra $(M, \cdot, \alpha)$ can be embedded into a multiplicative, weakly unital hom-associative algebra $(M \oplus R, \bullet, \beta_\alpha)$. For any $m_1, m_2 \in M$, $r_1, r_2 \in R$,

$$(m_1, r_1) \bullet (m_2, r_2) := (m_1 \cdot m_2 + r_1\alpha(m_2) + r_2\alpha(m_1), r_1r_2),$$
$$\beta_\alpha(m_1, r_1) := (\alpha(m_1), r_1).$$

Proposition ([BRS18])

$(M, \cdot, \alpha) \cong (M \oplus 0, \bullet, \beta_\alpha)$ *is a hom-ideal in* $(M \oplus R, \bullet, \beta_\alpha)$.

Proposition ([Yau09])

Let A be a unital, associative algebra with unit 1_A, α an algebra endomorphism on A, and define $*\colon A \times A \to A$ for all $a, b \in A$ by

$$a * b := \alpha(a \cdot b).$$

Then $(A, *, \alpha)$ is a weakly unital hom-associative algebra with weak unit 1_A.

[Yau09] D. Yau. "Hom-algebras and Homology". In: *J. Lie Theory* 19.2 (2009).

Definition (Hom-Lie algebra)

A *hom-Lie algebra* over an associative, commutative, and unital ring R is a triple $(M, [\cdot,\cdot], \alpha)$ where M is an R-module, $\alpha\colon M \to M$ a linear map, and $[\cdot,\cdot]\colon M \times M \to M$ a bilinear and alternative map, satisfying:

$$[\alpha(a),[b,c]] + [\alpha(c),[a,b]] + [\alpha(b),[c,a]] = 0, \quad \forall a,b,c \in M.$$

Proposition ([MS08])

Let $(M, \cdot, \alpha)$ *be a hom-associative algebra with commutator* $[\cdot,\cdot]$. *Then* $(M, [\cdot,\cdot], \alpha)$ *is a hom-Lie algebra.*

[MS08] A. Makhlouf and S.D. Silvestrov. “Hom-algebra structures”. In: *J. Gen. Lie Theory Appl.* 2.2 (2008).

Non-commutative, associative polynomial rings

Let R be an associative and unital ring, and consider $R[x]$ as an additive group. Want to make this an associative, non-commutative, unital ring S:

$\mathbf{deg}(p \cdot q) \leq \mathbf{deg}(p) + \mathbf{deg}(q)$ for any $p, q \in S$,

$x^m \cdot x^n = x^{m+n}$ for any $m, n \in \mathbb{N}$,

For any $a \in R$, we need $x \cdot a = \sigma(a)x + \delta(a)$ for some $\sigma, \delta \colon R \to R$ (while S is a left R-module). Iterating, we get

$$ax^m \cdot bx^n = \sum_{i \in \mathbb{N}} (a\pi_i^m(b))x^{i+n},$$

where $\pi_i^m \colon R \to R$ is the sum of all $\binom{m}{i}$ compositions of i copies of σ and $m - i$ copies of δ. For example, $\pi_1^2(b) = \sigma(\delta(b)) + \delta(\sigma(b))$.

Associative Ore extensions: σ and δ

S should be an associative and unital ring, so for any $a, b \in R$,

$$x \cdot (a+b) = x \cdot a + x \cdot b \quad \text{(left distributivity)},$$
$$x \cdot (ab) = (x \cdot a) \cdot b \quad \text{(associativity)},$$
$$x \cdot 1_R = 1_R \cdot x = x \quad \text{(unitality)}.$$

This implies

$$\sigma(1_R) = 1_R,$$
$$\sigma(a+b) = \sigma(a) + \sigma(b),$$
$$\sigma(ab) = \sigma(a)\sigma(b),$$

so σ needs to be an *endomorphism*. Moreover,

$$\delta(a+b) = \delta(a) + \delta(b),$$
$$\delta(ab) = \sigma(a)\delta(b) + \delta(a)b,$$

so δ is a σ-*derivation* (if $\sigma = \mathrm{id}_R$, a *derivation*). For such σ and δ we get an associative and unital ring $R[x; \sigma, \delta]$, the *Ore extension of R*.

Let R be an associative and unital ring, and $r \in R$.

Example (Polynomial ring)

A *polynomial ring* over R, written $R[x]$, is $R[x; \mathrm{id}_R, 0_R]$.
Here, $x \cdot r = rx$.

Example (Skew-polynomial ring)

A *skew-polynomial ring* over R is $R[x; \sigma, 0_R]$ for some endomorphism σ.
Here, $x \cdot r = \sigma(r)x$.

Example (Differential polynomial ring)

A *differential polynomial ring* over R is $R[x; \mathrm{id}_R, \delta]$, δ a derivation.
Here, $x \cdot r = rx + \delta(r)$.

In quantum mechanics, $x \cdot y - y \cdot x = i\hbar 1_{\mathbb{C}}$. The *Weyl algebra* A_1 over a field K, is $K\langle x, y\rangle / (x \cdot y - y \cdot x - 1_K)$. $A_1 = K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$.

Proposition (No zero divisors)

A_1 is a non-commutative domain.

Proposition (The center of A_1)

$$C(A_1) = \begin{cases} K & \textit{if } \operatorname{char}(K) = 0, \\ K[x^p, y^p] & \textit{if } \operatorname{char}(K) > 0. \end{cases}$$

Proposition (The derivations of A_1)

$$\mathrm{Der}_K(A_1) = \begin{cases} \mathrm{InnDer}_K(A_1) & \textit{if } \mathrm{char}(K) = 0, \\ C(A_1)E_x \oplus C(A_1)E_y \oplus \mathrm{InnDer}_K(A_1) & \textit{if } \mathrm{char}(K) > 0. \end{cases}$$

Here, $E_x, E_y \in \mathrm{Der}_K(A_1)$ *are defined by* $E_x(x) = y^{p-1}$, $E_x(y) = 0$, $E_y(x) = 0$, $E_y(y) = x^{p-1}$.

Conjecture ([Dix68])

When $\mathrm{char}(K) = 0$*, all endomorphisms on* A_1 *are automorphisms.*

[Dix68] J. Dixmier. "Sur les algèbres de Weyl". In: *Bull. Soc. Math. France* 96 (1968).

Non-commutative, hom-associative polynomial rings

Definition (Non-associative, non-unital Ore extension)

If R is a non-associative, non-unital ring, a map $\beta\colon R \to R$ is *left R-additive* if for all $r, s, t \in R$, $r \cdot \beta(s+t) = r \cdot (\beta(s) + \beta(t))$.

For σ and δ left R-additive maps on R, a *non-associative, non-unital Ore extension* of R, $R[x;\sigma,\delta]$, is the additive group $R[x]$ with

$$ax^m \cdot bx^n := \sum_{i\in\mathbb{N}} \left(a\pi_i^m(b)\right) x^{i+n}, \quad \forall a, b \in R.$$

If $\alpha\colon R \to R$ is any map, we may extend it *homogeneously* to $R[x;\sigma,\delta]$ by $\alpha(ax^m) := \alpha(a)x^m$.

Proposition ([BRS18])

Let R be a hom-associative ring with twisting map α, σ an endomorphism and δ a σ-derivation that both commute with α. Then $R[x;\sigma,\delta]$ is a hom-associative Ore extension, α extended homogeneously to $R[x;\sigma,\delta]$.

Proposition ([BRS18])

Let R be a unital, associative ring, σ an endomorphism, δ a σ-derivation, and α an endomorphism that commutes with σ and δ. Then $(R[x;\sigma,\delta],*,\alpha)$ is a weakly unital, hom-associative Ore extension, α extended homogeneously to $R[x;\sigma,\delta]$.

The above conditions turn out to be *almost* necessary as well.

The hom-associative Weyl algebras

Lemma ([BRS18], [BR20b])

Let K be a field and α an endomorphism on $K[y]$. Then α commutes with $\mathrm{d}/\mathrm{d}y$ if and only if

$$\alpha(y) = \begin{cases} k_0 + y & \text{if } \mathrm{char}(K) = 0, \\ k_0 + y + k_p y^p + k_{2p} y^{2p} + \ldots & \text{if } \mathrm{char}(K) > 0. \end{cases}$$

Here, $k_0, k_p, k_{2p}, \ldots \in K$.

Rename the above map α_k, $k := \begin{cases} k_0 & \text{if } \mathrm{char}(K) = 0, \\ (k_0, k_p, k_{2p}, \ldots) & \text{if } \mathrm{char}(K) > 0. \end{cases}$

Definition (The hom-associative Weyl algebras [BRS18], [BR20b])

The *hom-associative Weyl algebras* A_1^k are $(A_1, *, \alpha_k)$ where α_k is extended homogeneously to $A_1 = K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$.

If $k = 0$, then $\alpha_k = \mathrm{id}_{A_1}$, so $A_1^0 = A_1$. Also, $x * y - y * x = 1_K$. $1_K * y := \alpha_k(y)$, $\alpha_k(y) = y \iff k = 0$.

Proposition ([BR20a], [BR20b])

1_K is a unique weak unit in A_1^k.

A_1^k contain no zero divisors.

A_1^k is power associative if and only if $k = 0$.

$$N(A_1^k) = \begin{cases} A_1^k & \text{if } k = 0, \\ \{0\} & \text{otherwise.} \end{cases}$$

$$C(A_1^k) = C(A_1) = \begin{cases} K & \text{if } \mathrm{char}(K) = 0, \\ K[x^p, y^p] & \text{if } \mathrm{char}(K) > 0. \end{cases}$$

[BR20a] P. Bäck and J. Richter. "On the hom-associative Weyl algebras". In: *J. Pure Appl. Algebra* 224.9 (2020).

Proposition ([BR20a], [BR20b])

If $\mathrm{char}(K) = 0$, $k \neq 0$, *then* $\delta \in \mathrm{Der}_K(A_1^k)$ *if and only if* $\delta = \mathrm{ad}_{cy+q}$, $c \in K$, $q \in K[x]$.

If $\mathrm{char}(K) > 0$, $k = (k_0, k_p, k_{2p}, \ldots, k_{Mp}, 0, \ldots)$, $M \in \mathbb{N}_{>0}$, $k_{Mp} \neq 0$, *then* $\delta \in \mathrm{Der}_K(A_1^k)$ *if and only if*

$$\delta = \begin{cases} vE_y + \mathrm{ad}_r & \text{if } k = (k_0, 0, \ldots, 0, k_{p^2}, 0, \ldots, 0, k_{2p^2}, 0, \ldots), \\ \mathrm{ad}_r & \text{else.} \end{cases}$$

Here, $r = ayx + \sum_{i \equiv 0 \pmod p} b_i y^i x + c_i y x^i$ *for some* $a, b_i, c_i \in K$, $v \in K[x^p]$, *and* $E_y \in \mathrm{Der}_K(A_1)$ *is defined by* $E_y(x) = 0$, $E_y(y) = x^{p-1}$.

Proposition ([BR20a], [BR20b])

If $\mathrm{char}(K) = 0$, $k, l \neq 0$, *then any homomorphism* $f\colon A_1^k \to A_1^l$ *is an isomorphism* $f(x) = \frac{l}{k}x + c$, $f(y) = \frac{k}{l}y + q$, $c \in K$, $q \in K[x]$.

If $\mathrm{char}(K) > 0$, $k = (k_0, 0, \ldots) \neq 0$, $l = (l_0, 0, \ldots) \neq 0$, *then* $A_1^k \cong A_1^l$.

If $\mathrm{char}(K) > 0$, $k = (k_0, k_p, k_{2p}, \ldots, k_{Mp}, 0, \ldots)$, $l = (l_0, l_p, l_{2p}, \ldots, l_{Np}, 0, \ldots)$, $M, N \in \mathbb{N}_{>0}$, $k_{Mp}, l_{Np} \neq 0$, *then* $f\colon A_1^k \to A_1^l$ *is an isomorphism if and only if* $M = N$, $f \in \mathrm{Aut}_K(A_1)$ *with* $f(x) = b_0 + a_1^{-1}x$ *and* $f(y) = a_0 + a_1 y$, $a_0, b_0 \in K$, $a_1 \in K^\times$, *satisfying*

$$\sum_{i=j}^{M} \binom{i}{j} k_{ip} a_0^{(i-j)p} a_1^{jp-1} = l_{jp}, \quad 0 \leq j \leq M.$$

Corollary ([BR20a])

If $\mathrm{char}(K) = 0$, $k \neq 0$, *then any endomorphism* f *on* A_1^k *is an automorphism* $f(x) = x + c$ *and* $f(y) = y + q$, $c \in K$, $q \in K[x]$.

Multi-parameter formal hom-deformations

Definition (Multi-parameter formal hom-associative deformation)

An *n-parameter formal hom-associative deformation* of a hom-associative algebra over R, $(M, \cdot_0, \alpha_0)$, is a hom-associative algebra over $R[\![t_1, \ldots, t_n]\!]$, $(M[\![t_1, \ldots, t_n]\!], \cdot_t, \alpha_t)$, where

$$\cdot_t = \sum_{i \in \mathbb{N}^n} \cdot_i t^i, \quad \alpha_t = \sum_{i \in \mathbb{N}^n} \alpha_i t^i.$$

Here, $i := (i_1, \ldots, i_n)$, $t := (t_1, \ldots, t_n)$, and $t^i := t_1^{i_1} \cdots t_n^{i_n}$.

Proposition ([BR20a], [BR20b])

A_1^k are multi-parameter formal hom-associative deformations of A_1.

Remark

A_1 is rigid as an *associative* algebra when $\mathsf{char}(K) = 0$.

Definition (Multi-parameter formal hom-Lie deformation)

An *n-parameter formal hom-Lie deformation* of a hom-Lie algebra over R, $(M, [\cdot,\cdot]_0, \alpha_0)$, is a hom-Lie algebra over $R[\![t_1,\dots,t_n]\!]$, $(M[\![t_1,\dots,t_n]\!], [\cdot,\cdot]_t, \alpha_t)$, where

$$[\cdot,\cdot]_t = \sum_{i\in\mathbb{N}^n} [\cdot,\cdot]_i t^i, \quad \alpha_t = \sum_{i\in\mathbb{N}^n} \alpha_i t^i.$$

Here, $i := (i_1,\dots,i_n)$, $t := (t_1,\dots,t_n)$, and $t^i := t_1^{i_1}\cdots t_n^{i_n}$.

Proposition ([BR20a], [BR20b])

The hom-Lie algebras of A_1^k are multi-parameter formal hom-Lie deformations of the Lie algebras of A_1, using the commutator as bracket.

Thank you!