Noncommutative Minimal Surfaces

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References

The work I will present today is based on collaborations with Jaigyoung Choe, Jens Hoppe and Maxim Kontsevich:

Noncommutative Minimal Surfaces
J. A., J. Choe and J. Hoppe. Lett. Math. Phys. 2016.

Quantum Minimal Surfaces
J. A., J. Hoppe and M. Kontsevich. `arXiv:1903.10792`.

Why noncommutative minimal surfaces?

The classical theory of minimal surfaces is an old and rich subject, and still quite active.

From a mathematical point of view, it is interesting to investigate if one can develop a parallel theory in noncommutative geometry.

As I will show, we can construct many explicit examples of minimal surfaces that can be turned into noncommutative ones. In this way, one can provide a multitude of examples of noncommutative surfaces.

Analogues of minimal surface equations appear as equations of motion in physical models; e.g. in Membrane and String theory one finds that the (operators corresponding to the) embedding coordinates have to be "harmonic".

There are other approaches to noncommutative minimal embeddings (e.g. Dabrowski, Krajewski, Landi, Luef).

Minimal surfaces in Euclidean space

Let $\Omega \subseteq \mathbb{R}^2$ such that $\vec{x} : \Omega \to \mathbb{R}^n$ describes a surface Σ in $\mathbb{R}^n$.
Classically, $\vec{x} : \Omega \to \mathbb{R}^n$ is called a *minimal surface* if it is a stationary point of the area integral:

$$A[\vec{x}] = \int \sqrt{g}\, du dv$$

where g denotes the induced metric on Σ.This can be formulated as demanding that the embedding coordinates x^i are harmonic; i.e.

$$\Delta_\Sigma(x^i) = 0 \qquad \text{for } i = 1, 2, \ldots, n,$$

where Δ_Σ denotes the Laplace-Beltrami operator on Σ. (There are of course other characterizations.)

Poisson algebraic formulation of geometry

Assume that Σ is a 2-dimensional manifold, with local coordinates $u = u^1, v = u^2$, embedded in $\mathbb{R}^n$ via the embedding coordinates $x^1(u,v), x^2(u,v), \ldots, x^n(u,v)$, inducing on Σ the metric

$$g_{ab} = \partial_a \vec{x} \cdot \partial_b \vec{x} \equiv \sum_{i=1}^{n} \left(\partial_a x^i\right)\left(\partial_b x^i\right)$$

where $\partial_a = \frac{\partial}{\partial u^a}$. Indices a, b, p, q take values in $\{1, 2\}$, and i, j, k, l run from 1 to n.

For an arbitrary density ρ, one may introduce a Poisson bracket on $C^\infty(\Sigma)$ via

$$\{f, h\} = \sum_{a,b=1}^{2} \frac{1}{\rho} \varepsilon^{ab} \left(\partial_a f\right)\left(\partial_b h\right),$$

and we define the function $\gamma = \sqrt{g}/\rho$, where g denotes the determinant of the metric g_{ab}.

It turns out that one can formulate the Riemannian geometry of embedded almost (para-)Kähler manifolds in terms of the Poisson algebra of the manifold.

For instance, the Gaussian curvature of a surface embedded in $\mathbb{R}^n$ can be computed as

$$K = \sum_{j,k,l=1}^{n} \frac{1}{\gamma^4}\bigg(\frac{1}{2}\{\{x^j, x^k\}, x^k\}\{\{x^j, x^l\}, x^l\} - \frac{1}{4}\{\{x^j, x^k\}, x^l\}\{\{x^j, x^k\}, x^l\}\bigg).$$

where $x^1, \ldots, x^n$ denote the embedding coordinates.

Multilinear formulation of differential geometry and matrix regularizations
J. Diff. Geo. (J.A., J. Hoppe, G. Huisken, 2012)

Pseudo-Riemannian geometry in terms of multilinear brackets
Lett. Math. Phys. (J.A., G. Huisken, 2014)

In this spirit, one can show that the Laplace-Beltrami operator on Σ can be written in the following two forms

$$\Delta(f) = \gamma^{-1}\sum_{i=1}^{n}\{\gamma^{-1}\{f,x^i\},x^i\}$$
$$\Delta(f) = \gamma^{-1}\{\gamma^{-1}\{f,u^a\}g_{ab},u^b\}.$$

Note that, if $\gamma = \sqrt{g}/\rho = 1$ and $f = x^j$ one gets

$$\Delta(x^j) = \sum_{i=1}^{n}\{\{x^j,x^i\},x^i\}.$$

On a surface, one may always find *conformal coordinates*; i.e., coordinates with respect to which the metric becomes $g_{ab} = \mathcal{E}(u,v)\delta_{ab}$ for some (strictly positive) function $\mathcal{E}$. Furthermore, if we choose $\rho = 1$ (giving $\gamma = \mathcal{E}$), the formula above can be written as

$$\Delta(f) = \frac{1}{\mathcal{E}}\{\{f,u^a\}\delta_{ab},u^b\} = \frac{1}{\mathcal{E}}\{\{f,u\},u\} + \frac{1}{\mathcal{E}}\{\{f,v\},v\}$$

Minimal surfaces can be characterized by the fact that their embedding coordinates $x^1, \ldots, x^n$ are harmonic with respect to the Laplace operator on the surface; i.e. $\Delta(x^i) = 0$ for $i = 1, \ldots, n$. In local conformal coordinates, due to the above Poisson algebraic formulas, one may formulate this as follows:

A surface $\vec{x} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ is minimal if

$$\Delta_0(x^i) = \{\{x^i, u\}, u\} + \{\{x^i, v\}, v\} = 0 \text{ for } i = 1, \ldots, n$$
$$\vec{x}_u \cdot \vec{x}_u = \vec{x}_v \cdot \vec{x}_v \text{ and } \vec{x}_u \cdot \vec{x}_v = 0$$

We also note that the above Poisson bracket satisfies $\{u, v\} = 1$. These formulas make up our starting point when generalizing to noncommutative algebras.

To avoid an excess of notation and concepts in this talk, I will give a very basic presentation of the material. From an algebraic point of view, it can be made a lot more sophisticated. However, that is not the point of my talk today.

The Weyl algebra

In the geometrical setting, we introduced a Poisson bracket with $\{u, v\} = 1$. Therefore, we shall be interested in a (noncommutative) unital algebra generated by two elements U, V satisfying

$$[U, V] = i\hbar\mathbb{1},$$

for some real number $\hbar > 0$. The associative unital algebra generated by U, V satisfying the above relation is commonly referred to as the *Weyl algebra*.

The Weyl algebra satisfies the so called *Ore condition*, which implies that it can be embedded in a field of fractions by a general procedure. By $\mathcal{A}_\hbar$ we shall denote the Weyl algebra, and by $\mathfrak{F}_\hbar$ its field of fractions.

The Weyl algebra (and its field of fractions) can be equipped with a $*$-algebra structure by letting $U^* = U$ and $V^* = V$.

Derivations

Let us introduce the derivations

$$\hat{\partial}_u(A) \equiv \hat{\partial}_1(A) = \frac{1}{i\hbar}[A, V]$$
$$\hat{\partial}_v(A) \equiv \hat{\partial}_2(A) = -\frac{1}{i\hbar}[A, U],$$

from which it follows that $\hat{\partial}_u\big(\hat{\partial}_v(A)\big) = \hat{\partial}_v\big(\hat{\partial}_u(A)\big)$.

Compare with the geometric setting (with the choice $\rho = 1$), where it holds that $\frac{\partial f}{\partial u} = \{f, v\}$ and $\frac{\partial f}{\partial v} = -\{f, u\}$.

In analogy with complex analysis, we introduce

$$\Lambda = U + iV$$
$$\partial(A) = \frac{1}{2}\big(\hat{\partial}_u(A) - i\hat{\partial}_v(A)\big) = \frac{1}{2\hbar}[A, \Lambda^*]$$
$$\bar{\partial}(A) = \frac{1}{2}\big(\hat{\partial}_u(A) + i\hat{\partial}_v(A)\big) = -\frac{1}{2\hbar}[A, \Lambda],$$

and it follows that ∂ and $\bar{\partial}$ commute.

Harmonic elements

Let us define the Laplace operator

$$\Delta_0(A) = \hat{\partial}_u^2(A) + \hat{\partial}_v^2(A) = -\frac{1}{\hbar^2}\big[[A,V],V\big] - \frac{1}{\hbar^2}\big[[A,U],U\big],$$

in analogy with the classical expression

$$\{\{f,u\},u\} + \{\{f,v\},v\}.$$

An element $A \in \mathfrak{F}_\hbar$ is called *harmonic* if $\Delta_0(A) = 0$.
It also holds that

$$\Delta_0(A) = 4\partial\bar{\partial}(A) = 4\bar{\partial}\partial(A).$$

Moreover, we say that an element $A \in \mathfrak{F}_\hbar$ is *holomorphic* if $\bar{\partial}(A) = 0$. We also define

$$\int A = B \quad \text{if} \quad \partial B = A.$$

The noncommutative embedding

In analogy with the classical situation, where the embedding in $\mathbb{R}^n$ is given by the embedding coordinates $x^1, \dots, x^n$, we will think of an element of the free module (with basis $e_1, \dots, e_n$)

$$X = e_i X^i = e_1 X^1 + e_2 X^2 + \cdot + e_n X^n \in \mathfrak{F}_\hbar^n$$

for $X^1, \dots, X^n \in \mathfrak{F}_\hbar$, as representing a noncommutative embedding.

Moreover, for $X, Y \in \mathfrak{F}_\hbar^n$ one introduces the "Euclidean metric"

$$h(X, Y) = \sum_{i=1}^{n} (X^i)^* Y^i$$

and the derivations $\hat{\partial}_1, \hat{\partial}_2$ are extended to flat covariant derivatives

$$\hat{\partial}_a(X) = e_i \hat{\partial}_a(X^i)$$

for $a = 1, 2$.

Noncommutative Minimal Surfaces

Let us recall the formulation of a minimal surface that we would like to generalize to the noncommutative setting:

A surface $\vec{x} : D \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ is minimal if

$$\Delta_0(x^i) = \{\{x^i, u\}, u\} + \{\{x^i, v\}, v\} = 0 \text{ for } i = 1, \dots, n$$
$$\vec{x}_u \cdot \vec{x}_u = \vec{x}_v \cdot \vec{x}_v \text{ and } \vec{x}_u \cdot \vec{x}_v = 0$$

Let us turn this into a very naive definition.

Noncommutative Minimal Surfaces

Definition

An element $X = e_i X^i \in \mathfrak{F}_\hbar^n$ is called a *noncommutative minimal surface* if $(X^i)^* = X^i$ and

$$\Delta_0(X^i) = -\frac{1}{\hbar^2}\big[[X^i, V], V\big] - \frac{1}{\hbar^2}\big[[X^i, U], U\big] = 0$$
$$\mathcal{E} = \mathcal{G} \text{ and } \operatorname{Re}\mathcal{F} = 0,$$

for $i = 1, 2, \ldots, n$, where

$$\mathcal{E} = h\big(\hat{\partial}_u X, \hat{\partial}_u X\big) \quad \mathcal{G} = h\big(\hat{\partial}_v X, \hat{\partial}_v X\big),$$
$$\mathcal{F} = h\big(\hat{\partial}_u X, \hat{\partial}_v X\big)$$

The noncommutative Weierstrass representation theorem

The classical Weierstrass theorem gives a representation formula for *all* minimal surfaces in $\mathbb{R}^3$. It turns out that one can prove a nc analogue.

Theorem

Let $X = e_i X^i \in \mathfrak{F}_\hbar^3$ be a minimal surface for which it holds that $\partial(X^1 - iX^2) \neq 0$. Then there exist holomorphic elements $f, g \in \mathfrak{F}_\hbar$ together with $x^i \in \mathbb{R}$ (for $i = 1, 2, 3$), such that

$$\begin{aligned} X^1 &= x^1\mathbb{1} + \mathrm{Re}\int \frac{1}{2} f(\mathbb{1} - g^2)d\Lambda \\ X^2 &= x^2\mathbb{1} + \mathrm{Re}\int \frac{i}{2} f(\mathbb{1} + g^2)d\Lambda \\ X^3 &= x^3\mathbb{1} + \mathrm{Re}\int fg d\Lambda. \end{aligned} \tag{1}$$

Conversely, for any holomorphic f and g such that $f(1-g^2)$, $f(1+g^2)$ and fg are integrable, the above equations define a minimal surface.

Algebraic minimal surfaces

For instance, for arbitrary polynomial $F(\Lambda)$ the following defines a minimal surface in $\mathcal{A}^3_\hbar$:

$$X^1 = \mathrm{Re}\left((\mathbb{1} - \Lambda^2)\partial^2 F(\Lambda) + 2\Lambda\partial F(\Lambda) - 2F(\Lambda)\right)$$

$$X^2 = \mathrm{Re}\left(i(\mathbb{1} + \Lambda^2)\partial^2 F(\Lambda) - 2i\Lambda\partial F(\Lambda) + 2iF(\Lambda)\right)$$

$$X^3 = \mathrm{Re}\left(2\Lambda\partial^2 F(\Lambda) - 2\partial F(\Lambda)\right)$$

In other words, the above elements satisfy (for $i = 1, 2, 3$)

$$\big[[X^i, U], U\big] + \big[[X^i, V], V\big] = 0.$$

The simplest case is the noncommutative Enneper surface:

$$X^1 = U + UV^2 - \frac{1}{3}U^3 - i\hbar V$$

$$X^2 = -V - U^2 V + \frac{1}{3}V^3 + i\hbar U$$

$$X^3 = U^2 - V^2.$$

Double commutator equations

Recall that the Laplace operator can also be written as

$$\Delta(x^i) = \sum_{j=1}^{n} \{\{x^i, x^j\}, x^j\}.$$

in another choice of coordinates (where $\rho = \sqrt{g}$). The corresponding noncommutative equations for a minimal embedding

$$\sum_{j=1}^{n} \left[[X^i, X^j], X^j\right] = 0$$

are well-known in physics. (String theory, Membrane theory, ...) From a physical point of view it is important to find concrete solutions to these equations; i.e. operators satisfying the above equations. However, it turns out to be very hard to construct such solutions.

Quantum Minimal Surfaces

J. Arnlind, J. Hoppe, M. Kontsevich. arXiv:1903.10792

In the previous work, we found many explicit solutions to another set of "double-commutator" equations that, in the classical case, are related to these by a change of coordinates. Hence, solutions of one set of equations give rise to solutions of the other set of equations.

Is there an analogous way to construct solutions to

$$\sum_{j=1}^{n} \big[[X^i, X^j], X^j\big] = 0$$

from solutions to

$$\big[[X^i, U], U\big] + \big[[X^i, V], V\big] = 0?$$

The answer in general is most likely "no", but we managed to implement a noncommutative coordinate change for certain operator representations to construct, for instance, a solution related to the catenoid.

In short, one starts from a Fock-space representation of the operators $W = X^1 + iY^2$ and X^3 with the Ansatz

$$W\big|n\rangle = w_n\big|n-1\rangle \qquad X^3\big|n\rangle = z_n\big|n\rangle$$

and derive coupled recursion relations for the coefficients $\{w_n, z_n\}$. With some effort (and discovering connections to discrete integrable systems), we were able to prove the existence of solutions to these recursion relations, giving three operators X^1, X^2, X^3 satisfying

$$\sum_{j=1}^{n}\big[[X^i, X^j], X^j\big] = 0.$$

Thank you!