Formal hom-associative deformations of Ore extensions

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Per Bäck, <per.back@mdh.se>
Mälardalen University, Sweden

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Introduction

Hom-associative algebras – algebras with the associativity condition twisted by a *hom*omorphism – arose with hom-Lie algebras, introduced by Hartwig, Larsson, and Silvestrov [HLS06].

Non-commutative polynomial rings – or *Ore extensions* – were introduced by Ore Ore33, and generalized to non-associative such by Nystedt, Öinert, and Richter [NÖR18].

Naïve idea – why not *hom-associative Ore extensions*?

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Hom-associative algebras: Preliminaries

Definition (Hom-everything)

A *hom-associative algebra* over an associative, commutative, and unital ring R, is a triple $(M, \cdot, \alpha)$ consisting of an R-module M, a binary operation $\cdot\colon M \times M \to M$ linear over R in both arguments, and an R-linear map $\alpha\colon M \to M$, satisfying, for all $a, b, c \in M$,

$$\alpha(a) \cdot (b \cdot c) = (a \cdot b) \cdot \alpha(c).$$

A *hom-associative ring* is a hom-associative algebra over $\mathbb{Z}$.

A map $f\colon A \to B$ between hom-associative algebras is a *homomorphism* if it is linear, multiplicative, and $f \circ \alpha_A = \alpha_B \circ f$.

A left (right) ideal I s.t. $\alpha(I) \subseteq I$ is a left (right) *hom-ideal*.

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Proposition ([BRS18])
Any multiplicative hom-associative R-algebra $(M, \cdot, \alpha)$ can be embedded into a multiplicative, weakly unital hom-associative algebra $(M \oplus R, \bullet, \beta_\alpha)$. For any $m_1, m_2 \in M$, $r_1, r_2 \in R$,

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Proposition ([Yau09])

Let A be a unital, associative algebra with unit 1_A, α an algebra endomorphism on A, and define $*\colon A \times A \to A$ for all $a, b \in A$ by

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$$\big[\alpha(a), [b,c]\big] + \big[\alpha(c), [a,b]\big] + \big[\alpha(b), [c,a]\big] = 0.$$

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Let $(M, \cdot, \alpha)$ be a hom-associative algebra with commutator $[\cdot,\cdot]$. Then $(M, [\cdot,\cdot], \alpha)$ is a hom-Lie algebra.

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Definition (Left R-additivity)

If R is a non-associative, non-unital ring, a map $\beta\colon R\to R$ is *left R-additive* if for all $r,s,t\in R$, $r\cdot\beta(s+t)=r\cdot\big(\beta(s)+\beta(t)\big)$.

If $\delta\colon R\to R$ and $\sigma\colon R\to R$ are left R-additive maps, by a *non-associative, non-unital Ore extension* of R, $R[x;\sigma,\delta]$, we mean $\left\{\sum_{i\in\mathbb{N}} a_i x^i\right\}$, finitely many $a_i\in R$ non-zero, endowed with the addition

$$\sum_{i\in\mathbb{N}} a_i x^i+\sum_{i\in\mathbb{N}} b_i x^i := \sum_{i\in\mathbb{N}}(a_i+b_i)x^i,\quad a_i,b_i\in R,$$

two polynomials being equal iff their coefficients are, $\forall a,b\in R$,

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For instance,

$$ax^0 \cdot bx^0 = \sum_{i\in\mathbb{N}} \big(a \cdot \pi_i^0(b)\big)x^{i+0} = (a\cdot b)x^0, \text{ so } R \cong Rx^0,$$
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Definition (σ-derivation)

A map $\delta\colon R \to R$ is a *σ-derivation* if $\delta(a\cdot b) = \delta(a)\cdot b + \sigma(a)\cdot\delta(b)$, $a, b \in R$, σ an endomorphism. If $\sigma = \mathrm{id}_R$, δ is a *derivation*.

If $\alpha\colon R \to R$ is any map, we may extend it *homogeneously* to $R[x;\sigma,\delta]$ by $\alpha(ax^m) := \alpha(a)x^m$.

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Hom-assoc. Ore extensions: Necessity

Proposition ([BRS18])

Let $R[x;\sigma,\delta]$ be a non-unital, hom-associative Ore extension of a non-unital, hom-associativ ring R with twisting map $\alpha\colon R\to R$, extended homogeneously to $R[x;\sigma,\delta]$. Then, for all $a,b,c\in R$,

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(a\cdot b)\cdot\delta(\alpha(c)) &= (a\cdot b)\cdot\alpha(\delta(c)),\\
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Hom-assoc. Ore extensions: Sufficiency

Proposition ([BRS18])

Assume α is the twisting map of a non-unital, hom-associative ring R, and extend the map homogeneously to $R[X;\sigma,\delta]$. Assume α commutes with δ and σ, and that σ is an endomorphism and δ a σ-derivation. Then $R[X;\sigma,\delta]$ is hom-associative.

Proposition ([BRS18])

Let $R[x;\sigma,\delta]$ be a non-unital, associative Ore extension of a non-unital, associative ring R, where σ is an endomorphism and δ a σ-derivation. Assume α is a ring endomorphism that commutes with σ and δ. Then $\big(R[x;\sigma,\delta],*,\alpha\big)$ is a multiplicative, non-unital, hom-associative Ore extension with α extended homogeneously to $R[x;\sigma,\delta]$.

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Hom-assoc. Ore extensions: Examples

K a field, $\mathrm{char}(K) = 0$.

Example

The (associative) *quantum plane* $\mathcal{O}_q(K^2)$ is $K\langle x, y\rangle / (x \cdot y - qy \cdot x)$, $q \in K^\times$. $\mathcal{O}_q(K^2) \cong K[y][x; \sigma_q, 0_{K[y]}]$ where $\sigma_q(y) := qy$.

The *hom-associative quantum planes* $\mathcal{O}_q^k(K^2)$ are $(\mathcal{O}_q(K^2), *, \alpha_k)$ where, $\alpha_k(y) := ky$, and $\alpha_k(x) := x$ for $k \in K^\times$. Here, $x * y = kqy * x$ while $x * (y * y) - (x * y) * y = (k-1)k^3q^2y^2x$.

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$U(L)$ the universal enveloping algebra of the two-dimensional, non-abelian Lie algebra L defined by $[x, y] = y$. $U(L) \cong K[y][x; \mathrm{id}_{K[y]}, y\mathrm{d}/\mathrm{d}y]$.

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The hom-associative Weyl algebras

Example

In Quantum Mechanics, $p \cdot q - q \cdot p = i\hbar 1$ (or $p \cdot q - q \cdot p = 1$). The first (associative) *Weyl algebra* $A_1(K)$ is $K\langle x, y\rangle / (x \cdot y - y \cdot x - 1_K)$, $A_1(K) \cong K[y][x; \mathrm{id}_{K[y]}, \mathrm{d}/\mathrm{d}y]$.

Conjecture ([Dix68]): All endomorphisms on $A_1(K)$ are automorphisms.

The *hom-associative Weyl algebras* $A_1^k(K)$ are $(A_1(K), *, \alpha_k)$ where $\alpha_k(y) := y + k$, and $\alpha_k(x) := x$ for $k \in K$. Here, $[x, y]_* := x * y - y * x = 1_K$, while $1_K * y = \alpha_k(y) = y + k$.

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Representations of $A_1^k(K)$

$f: A_1(K) \to A_1'(K) \subset M_\infty(K)$ by

$$x \mapsto X := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad y \mapsto Y := \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$[X, Y] = I$. Define $\alpha_k'(X) := X$, $\alpha_k'(Y) := Y + kI$. Then $A_1^k(K) := (A_1(K), *, \alpha_k) \cong (A_1'(K), *', \alpha_k')$.

$g: A_1(K) \to A_1''(K) \subset \mathrm{End}_K(K[z])$ by

$$x \mapsto D_z := \mathrm{d}/\mathrm{d}z, \quad y \mapsto M_z := z\,\mathrm{id}_{K[z]},$$

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$$x \mapsto X := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \quad y \mapsto Y := \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

$[X, Y] = I$. Define $\alpha_k'(X) := X$, $\alpha_k'(Y) := Y + kI$. Then $A_1^k(K) := (A_1(K), *, \alpha_k) \cong (A_1'(K), *', \alpha_k')$.

$g\colon A_1(K) \to A_1''(K) \subset \mathrm{End}_K(K[z])$ by

$$x \mapsto D_z := \mathrm{d}/\mathrm{d}z, \quad y \mapsto M_z := z\,\mathrm{id}_{K[z]},$$

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Proposition ([BR19])

- $\alpha_k = e^{k\frac{\partial}{\partial y}}$, *so for all* $p, q \in A_1^k(K)$, $p * q = e^{k\frac{\partial}{\partial y}}(p \cdot q)$.
- $A_1^k(K)$ *is simple and contains no zero divisors.*
- $A_1^k(K)$ *is power associative if and only if* $k = 0$.
- $C(A_1^k(K)) = K$.
- $Z(A_1^k(K)) = \begin{cases} K & \text{if } k = 0, \\ \{0\} & \text{otherwise.} \end{cases}$

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Any homomorphism $f\colon A_1^k(K) \to A_1^l(K)$ for $k, l \neq 0$ is an isomorphism with $f(x) = \frac{l}{k}x + c$, $f(y) = \frac{k}{l}y + p(x)$ for $c \in K$ and $p(x) \in K[x]$.

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Formal hom-associative deformations

Definition (One-parameter formal hom-associative deformation)

A *one-parameter formal hom-associative deformation* of a hom-associative algebra over R, $(M, \cdot_0, \alpha_0)$ is a hom-associative algebra over $R[[t]]$, $(M[[t]], \cdot_t, \alpha_t)$, where

$$\cdot_t = \sum_{i \in \mathbb{N}} \cdot_i t^i, \quad \alpha_t = \sum_{i \in \mathbb{N}} \alpha_i t^i.$$

Proposition ([BR19], [Bäc19])

$A_1^k(K)$, $\mathcal{O}_q^k(K^2)$, and $U^k(L)$ are one-parameter formal hom-associative deformations of $A_1(K)$, $\mathcal{O}_q(K^2)$, and $U(L)$.

Remark

$A_1(K)$ is formally rigid as an *associative* algebra.

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The hom-Lie algebras of $A_1^k(K)$, $\mathcal{O}_q^k(K^2)$, and $U^k(L)$ are one-parameter formal hom-Lie deformations of the Lie algebras of $A_1(K)$, $\mathcal{O}_q(K^2)$, and $U(L)$, using the commutator as bracket.

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Thank you!