Projectors on the noncommutative cylinder

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References

Projections, modules and connections for the noncommutative cylinder
J.A. and G. Landi. `arXiv:1901.07276`

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- We thought that it would be interesting to really see the differences.
- In particular, we were interested in projective modules over the noncommutative cylinder.
- Let me give you an introduction to the noncommutative cylinder, as well as an explicit construction of projections representing all classes in K-theory.

The noncommutative cylinder

The noncommutative cylinder can be defined via a twisted convolution product a la Rieffel (with respect to a cocycle), but let us present it as follows. Let $\mathcal{S}(\mathbb{R} \times S^1)$ denote the space of Schwartz functions on $\mathbb{R} \times S^1$. Every $f \in \mathcal{S}(\mathbb{R} \times S^1)$ may be written as

$$f(u,t) = \sum_{n \in \mathbb{Z}} f_n(u) e^{2\pi i n t} \tag{1}$$

with $f_n \in \mathcal{S}(\mathbb{R})$.

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$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u) e^{2\pi i n t} \quad \text{and} \quad g(u,t) = \sum_{n\in\mathbb{Z}} g_n(u) e^{2\pi i n t}$$

we define

$$(f \bullet_\hbar g)(u,t) = \sum_{n\in\mathbb{Z}} \left[\sum_{k\in\mathbb{Z}} f_k(u) g_{n-k}(u + k\hbar) \right] e^{2\pi i n t}.$$

The noncommutative cylinder

Denote $W = e^{2\pi i t}$ (which is strictly speaking *not* in the algebra, since it does not decay) and note that we can think of the product in the algebra as functions of u commuting and the commutation with W as a shift; i.e.

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The noncommutative cylinder was studied by W. van Suijlekom (JMP, 2004), but a particular cocycle was not chosen giving the formulas we present above. Furthermore, no study of derivations,traces or projective modules was initiated. He did however compute the K-theory ($K_0 = \mathbb{Z}$), which I will come back to.

Derivations

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Derivations

The algebra I've presented can be completed in to a C^*-algebra $\mathcal{C}_\hbar$ in a standard way, but we will be mostly interested in the smooth part $\mathcal{C}_\hbar^\infty$. There are two canonical derivations on $\mathcal{C}_\hbar^\infty$. For

$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u) W^n$$

define

$$\partial_1 f = \sum_{n\in\mathbb{Z}} f_n'(u) W^n \quad \text{and} \quad \partial_2 f = 2\pi i \sum_{n\in\mathbb{Z}} n f_n(u) W^n.$$

Then ∂_1 and ∂_2 are hermitian derivations of $\mathcal{C}_\hbar^\infty$ and $[\partial_1, \partial_2] = 0$.

Trace / Integral

For $f \in \mathcal{C}^\infty_\hbar$ with

$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u) W^n$$

we set (note that Schwartz functions are integrable)

$$\tau(f) = \int_{-\infty}^{\infty} f_0(u) du.$$

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τ is a positive invariant trace; that is, it has the properties

1. $\tau(f^*) = \overline{\tau(f)}$,
2. $\tau(f^* f) \geq 0$,
3. $\tau(fg) = \tau(gf)$,
4. $\tau(\partial_1 f) = \tau(\partial_2 f) = 0$,

for all $f, g \in \mathcal{C}_\hbar^\infty$.

Projective modules

Let us consider a projective module defined by a projection. A projection $p \in M_n(A)$ (i.e a $n \times n$ matrix over the algebra A satisfying $p^2 = p$) defines a projective module as its image when acting on a free module of rank n as a matrix:

$$p(v) = p\Big(\sum_{i=1}^{n} e_i v^i\Big) = \sum_{i,j=1}^{n} e_i p^i_j v^j$$

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Finitely generated projective modules are therefore considered to be the "vector bundles" of noncommutative geometry.

K-theory

Essence of K_0:

- For an algebra A, $K_0(A)$ is the set of equivalence classes of projections in $M_\infty(A)$.

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K-theory (including higher K-groups) is invariant under Morita equivalence.

Projectors in the algebra

Can one find projectors in the algebra itself (i.e. a (1×1)-matrix)? An algebra element such that $p^* = p$ and $p^2 = p$?

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On a connected manifold, there are no nontrivial continuous functions f such that $f^2 = f$ (i.e. only $f = 1$ and $f = 0$).

However, in noncommutative geometry, there might be nontrivial projections in the algebra itself. A well-known case is the non-commutative torus.

Can one find projectors on the noncommutative cylinder?

Projections in $\mathcal{C}_\hbar^\infty$

Let us make the following Ansatz for a projection:

$$p = g(u+\hbar)W + f(u) + g(u)W^{-1}.$$

with f, g being real-valued (note that $p^* = p$ by construction).

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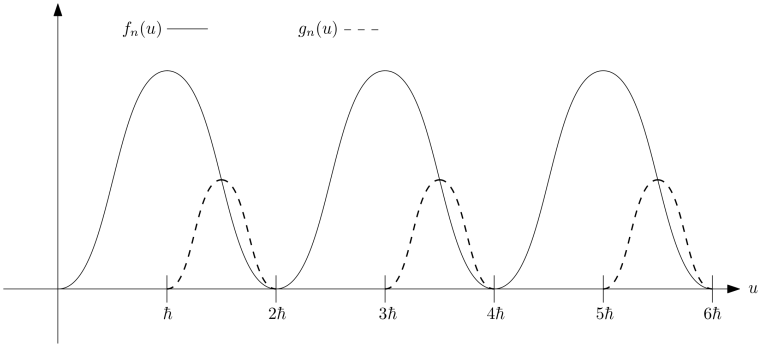
$$\begin{aligned}
&g(u)g(u+\hbar) = 0 \\
&g(u)\big(1 - f(u) - f(u-\hbar)\big) = 0 \\
&g(u)^2 + g(u+\hbar)^2 = f(u) - f(u)^2
\end{aligned}$$

We can find functions f, g satisfying these equations.

Projections in $\mathcal{C}_\hbar^\infty$

f and g can be given as any number of repetitions of functions with support in $[0, 2\hbar]$ as in the following figure:



$$g(u) = \sqrt{f(u) - f(u)^2} \qquad \text{(when } g(u) \neq 0\text{)}$$

A class of projective modules

Hence, for every integer $n \geq 1$, we have constructed a projection $p_n \in \mathcal{C}_\hbar^\infty$ as

$$p_n = g_n(u+\hbar)W + f_n(u) + g_n(u)W^{-1},$$

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The above result shows that the modules M_n and M_m are equivalent if and only if $n = m$, since if two projections p and q are equivalent in a C^*-algebra A then there exists $u \in A$ such that $p = uqu^{-1}$ implying that

$$\mathrm{tr}(p) = \mathrm{tr}(uqu^{-1}) = \mathrm{tr}(u^{-1}uq) = \mathrm{tr}(q)$$

Representatives of $K_0(\mathcal{C}_\hbar)$

Next, let us show that these projections respect the group structure of $\mathbb{Z}$.

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Let n, m be integers with $n, m \geq 1$. Then

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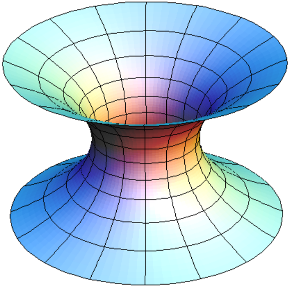
The proof is done by "pasting" the functions f_n, g_n and f_m, g_m next to each other. One has to prove that the projective module defined by shifted functions is equivalent to the unshifted module.

Hence, the group generated by the projective modules is isomorphic to $\mathbb{Z}$, giving representatives of the classes of $K_0(\mathcal{C}_\hbar)$.

A noncommutative catenoid

Let us do some Riemannian geometry of the noncommutative cylinder in the form of a noncommutative catenoid.



The catenoid is a minimal surface in $\mathbb{R}^3$. It has the topology of a cylinder, but the induced metric from $\mathbb{R}^3$ is not flat.

The catenoid

Let $\widehat{\mathcal{C}}_{\hbar}^{\infty}$ denote a slightly different algebra; namely, we consider elements of the form

$$f(u,t) = \sum_{n\in\mathbb{Z}} f_n(u) e^{2\pi int}$$

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Let $\mathfrak{g}$ denote the (abelian) Lie algebra generated by the derivations ∂_1 and ∂_2, and let $M = (\widehat{\mathcal{C}}^{\infty}_{\hbar})^2$ be a free module with basis e_1, e_2. Elements of M correspond to noncommutative "vector fields".

A metric on M is given by a invertible hermitian form $h : M \to M \to \mathcal{A}$, determined by

$$h_{ij} = h(e_i, e_j)$$

An affine connection $\nabla : \mathfrak{g} \times M \to M$ is metric if

$$\partial\big(h(U, V)\big) = h\big(\nabla_\partial U, V\big) + h\big(U, \nabla_\partial V\big)$$

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A metric and torsion-free real connection is called a Levi-Civita connection. In the setting of pseudo-Riemannian calculi (see Axel's talk) there exists a unique Levi-Civita connection on the noncommutative cylinder (for any metric).

Connection and curvature

For $h_{ij} = e^{2k(u)}\delta_{ij}$ one obtains

$$\nabla_1 e_1 = e_1 k'(u) \qquad \nabla_1 e_2 = \nabla_2 e_1 = e_2 k'(u) \qquad \nabla_2 e_2 = -e_1 k'(u).$$

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and

$$\begin{aligned}
R(\partial_1, \partial_2)e_1 &= \nabla_1\nabla_2 e_1 - \nabla_2\nabla_1 e_1 = e_2 k''(u) \\
R(\partial_1, \partial_2)e_2 &= \nabla_1\nabla_2 e_2 - \nabla_2\nabla_1 e_2 = -e_1 k''(u) \\
R_{1212} &= h\big(e_1, R(\partial_1, \partial_2)e_2\big) = -e^{2k(u)} k''(u),
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\end{aligned}$$

giving the Gaussian curvature as

$$K = \frac{1}{2}h^{ij}R_{ikjl}h^{kl} = -e^{-2k(u)}k''(u).$$

For a metric of the above form, a natural integration measure corresponding to the volume form is given by $\tau_h(f) = \tau(fe^{2k(u)})$. The total curvature is then

$$\tau_h(K) = -\int_{-\infty}^{\infty} e^{-2k(u)} k''(u) e^{2k(u)} du = -\int_{-\infty}^{\infty} k''(u) du$$
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Here one notes a certain independence of the total curvature with respect to perturbations of the metric; i.e. for $\tilde{k}(u) = \delta(u) + k(u)$ one finds that $\tau_h(\tilde{K}) = \tau_h(K)$ whenever

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For instance, for $k(u) = \ln(\cosh(u))$, corresponding to the induced metric on the catenoid, one obtains

$$\tau_h(K) = \lim_{u \to -\infty} \tanh(u) - \lim_{u \to \infty} \tanh(u) = -2,$$

valid also for all pertubations of the metric as above.

Thanks for listening!