Geometric aspects of noncommutative principal bundles

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Definition (Free actions on C*-algebras)

An action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is called *free* if the *Ellwood map*

$$\Phi : \mathcal{A} \otimes_{\mathrm{alg}} \mathcal{A} \to C(G, \mathcal{A}), \quad \Phi(x \otimes y)(g) := x\alpha_g(y)$$

has dense range.

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Remark (Smooth principal bundles)

In the smooth category there is a bijective correspondence between free (and proper) group actions and *locally trivial principal bundles*.

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We point out that $\mathbb{T}^2_\theta \cong C(\mathbb{T}^2)\,(\triangleq \mathbb{T}^2)$ if and only if $\theta \in \mathbb{Z}$. Moreover, the map $\alpha : \mathbb{T}^2 \to \mathrm{Aut}(\mathbb{T}^2_\theta)$ given on generators by

$$\alpha_{(z,w)}(U) := z \cdot U \quad \text{and} \quad \alpha_{(z,w)}(V) := w \cdot V$$

is a free and ergodic action of $\mathbb{T}^2$ on $\mathbb{T}^2_\theta$.

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is a free action of $\mathbb{T}$ on $\mathrm{SU}_q(2)$. It is called the *quantum Hopf fibration*.

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  - Their applications in T-duality may lead to a better understanding of T-duals and the question of their existence.
  - They may be used to develop and a theory of *quantum gerbes* and a *fundamental group* for noncommutative spaces (cf. [1]).

2. Classification of noncommutative principal bundles

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Remark (Classification of smooth principal bundles)

Given a smooth manifold M and a Lie group G, *Čech cohomology* provides a method for classifying smooth principal G-bundles over M. In fact,

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Problem (Classification of free actions)

Given a unital C*-algebra $\mathcal{B}\,(\hat{=}\; C(M))$ and a compact group G, understand and classify all free actions $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ such that $\mathcal{A}^G = \mathcal{B}$.

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- Each $A(\pi)$, $\pi \in \hat{G}$, carries a natural Hilbert $\mathcal{B}$-module structure w. r. t.

$$\langle x, y \rangle_{\mathcal{B}} := P_0(x^* y) := \int_G \alpha_g(x^* y)\, dg, \quad x, y \in A(\pi).$$

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- The multiplication between isotypic components is well captured by family of maps (fusion rules)

$$m_{\pi,\rho} : A(\pi) \otimes_{\mathcal{B}} A(\rho) \longrightarrow A(\pi \otimes \rho), \quad m_{\pi,\rho}(x \otimes y) := x \cdot y.$$

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- For free actions the fusion rules are particularly good-natured which makes classification certainly (more) available.

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- For each representation π of G we define the *-homomorphism

$$\gamma_\pi : \mathcal{B} \to \mathcal{L}(\mathcal{H}_\pi) \otimes \mathcal{B}, \quad \gamma_\pi(b) := s(\pi)^* \, (\mathbb{1}_{V_\pi} \otimes b) \, s(\pi)$$

  and for each pair π, ρ of representations of G an element

$$\omega(\pi, \rho) := s(\pi \otimes \rho)^* \, s(\pi) \, s(\rho) \in \mathcal{L}(\mathcal{H}_\pi \otimes \mathcal{H}_\rho, \mathcal{H}_{\pi \otimes \rho}) \otimes \mathcal{B}.$$

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- The corresponding collection $(\mathcal{H}, \gamma, \omega) = \big(\mathcal{H}_\pi, \gamma_\pi, \omega(\pi, \rho)\big)_{\pi,\rho \in \hat{G}}$ is called *a factor system* of $\alpha : G \to \mathrm{Aut}(\mathcal{A})$.

Factor systems are the key feature in our research program. In fact, they satisfy interesting algebraic relations that make free actions accessible to classification, K-theoretic considerations, and computations in general.

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Theorem (Schwieger-W. 15',16',17')

Let $\mathcal{B}$ be a unital C*-algebra and G a compact group. In [2–4] we provided a complete classification of free actions of G with fixed point algebra $\mathcal{B}$ in terms of *Hilbert $\mathcal{B}$-modules* and *factor systems*.

3. Geometric aspects of noncommutative principal bundles

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Remark (The Athiya sequence)

Given a principal G-bundle $q : P \to M$, connection 1-forms on P are in a $1:1$-correspondence with $C^\infty(M)$-linear sections of the *Athiya-Sequence*

$$0 \longrightarrow \mathfrak{gau}(P) \longrightarrow \mathcal{V}(P)^G \longrightarrow \mathcal{V}(M) \longrightarrow 0.$$

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Heuristic noncommutative approach:

Given a free action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ with fixed point algebra $\mathcal{B}$, study its geometric aspects in terms of a "*generalized Athiya sequence*"

$$\mathrm{der}_G(A) \longrightarrow \mathrm{der}(B), \quad \delta \mapsto \delta_{|\mathcal{B}}.$$

The main challenge of this approach is to find suitable conditions that help to decide whether a given *-derivation on "$\mathcal{B}$" extends to a G-equivariant *-derivation on "$\mathcal{A}$".

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(ii) a way to extend a given *-derivation $\delta_{\mathcal{B}} : \mathcal{B}_0 \to \mathcal{B}_0$ to a G-equivariant *-derivation $\delta_{\mathcal{A}} : \mathcal{A}_0 \to \mathcal{A}_0$.

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- Our results may be used to transfer the notions of connection 1-forms, connections, parallel transport, curvature, and characteristic classes to the noncommutative setting.
- The mathematical description for classical gauge theories is given in terms of smooth principal bundles. Hence, our analysis could yield a natural framework for studying noncommutative gauge theories.

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Definition (Cleft actions)

An action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ is called *cleft* if there exists a unitary element $u \in M(G) \otimes \mathcal{A}$ satisfying $\alpha_g(u) = \lambda_g^* u$ for all $g \in G$.

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- Each cleft action is free, but the converse does not hold. For instance, the quantum Hopf fibration is not cleft.
- Cleft means that the coisometries discussed before are in fact unitaries and the element $u \in M(G) \otimes \mathcal{A}$ is just the collection of all u_π, $\pi \in \hat{G}$.

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Theorem (W. 18')

Let $\mathcal{B}_0$ be a dense unital *-subalgebra of $\mathcal{B}$. Moreover, let $\delta_{\mathcal{B}} : \mathcal{B}_0 \to \mathcal{B}_0$ be a *-derivation. Then the following assertions hold:

In what follows we fix a cleft action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ and put $\mathcal{B} := \mathcal{A}^G$. Moreover, we choose a unitary $u \in M(G) \otimes \mathcal{A}$ with $\alpha_g(u) = \lambda_g^* u$ for all $g \in G$ and let $(\mathcal{H}, \gamma, \omega)$ be the corresponding factor system.

Theorem (W. 18')

Let $\mathcal{B}_0$ be a dense unital *-subalgebra of $\mathcal{B}$. Moreover, let $\delta_{\mathcal{B}} : \mathcal{B}_0 \to \mathcal{B}_0$ be a *-derivation. Then the following assertions hold:

(a) The set

$$\mathcal{A}_0 := \{\mathrm{Tr}(ux) \mid x \in M_0(G) \otimes \mathcal{B}_0\}$$

gives a G-invariant, dense unital *-subalgebra of $\mathcal{A}$ with $\mathcal{A}_0 \cap \mathcal{B} = \mathcal{B}_0$.

Theorem continued (W. 18')

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$$\gamma\delta_{\mathcal{B}}(b) - \delta_{\mathcal{B}}\gamma(b) = \imath[H, \gamma(b)], \quad \forall b \in \mathcal{B}_0, \tag{1}$$
$$-\imath\omega^*\delta_{\mathcal{B}}(\omega) = \mathrm{id} \otimes\gamma(H) + H_2 - \omega^*\Delta(H)\omega. \tag{2}$$

(c) If $H \in M(G) \otimes \mathcal{B}_0$ is self-adjoint and satisfies (1) and (2), then

$$\delta_{\mathcal{A}}\big(\mathrm{Tr}(ux)\big) := \mathrm{Tr}\big(u\delta_{\mathcal{B}}(x) + \imath uHx\big), \quad x \in M_0(G) \otimes \mathcal{B}_0,$$

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- Investigate related notions such as *connections*, *parallel transport*, *curvature*, and *characteristic classes*.

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