Epsilon-strong systems
skew inverse semigroup rings
and Steinberg algebras

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Steinberg algebras

Studied by a lot of people...

Ara, Arando Pino, Barquero, Beuter, Brown, Carlsen, Clark, Demeneghi, Edie-Michell, Exel, Farthing, Goncalez, Goncalves, Hazrat, an Huef, Li, Molina, Nam, Öinert, Pangalela, Pardo, Raeburn, Rout, Royer, Ruiz, Sierakowski, Siles Molina, Sims, Steinberg, Tomforde...

and many more!

Why?

• Old question by von Neumann (and others): ”What properties in operator algebra theory arise from discrete underlying strutures?”

• Algebraic analogues of groupoid C*-algebras.

• Includes discrete inverse semigroup algebras and the class of higher rank Kumjian-Pask algebras which in turn includes the class of Leavitt path algebras.

Theorem A

(Brown, Steinberg, Clark and Edie-Michel 2016)

If G is a Hausdorff and ample topological groupoid, and K is a commutative unital ring, then the Steinberg algebra $A_K(G)$ is simple if and only if G is effective and minimal, and K is a field.

Theorem B

(Beuter, Goncalves, Öinert and Royer 2017)

If π is a locally unital partial action of an inverse semigroup S on an associative, commutative and locally unital ring A, then the skew inverse semigroup ring $A \rtimes_\pi S$ is simple if and only if A is S-simple and A is a maximal commutative subring of $A \rtimes_\pi S$.

Rosetta stone

From a result by Beuter and Goncalves (2017) it follows that every Steinberg algebra can be described as a skew inverse semigroup ring.

Theorem

(Beuter, Goncalves, Öinert and Royer 2017)

Theorem B $\Rightarrow$ Theorem A

My motivation

(Remark by Beuter and Goncalves 2017)

Let $R = A \rtimes_\pi S$ **and put** $R_s = \overline{D_s \delta_s}$**.**

- R **is a system:**

$R = \sum_{s \in S} R_s$ **and** $R_s R_t \subseteq R_{st}$ **for** $s, t \in S$**.**

- R **is coherent:**

$R_s \subseteq R_t$ **for** $s, t \in S$ **with** $s \leq t$**.**

My question

Is there a generalization of Theorem B, valid for coherent systems?
Yes, at least the sufficient parts...

Theorem C

If S **is an inverse semigroup and** R **is a system simple coherent left (right) s-unital epsilon-strong system and** $C_R(Z(R_0)) \subseteq R_0$**, then** R **is simple.**

Theorem D

If R **is an idempotent coherent, system simple, left (right) minimally nondegenerate system and** $C_R(Z(R_0)) \subseteq R_0$**, then** R **is simple.**

Three implications

Theorem D

$$\Downarrow$$

Theorem C

$$\Downarrow$$

(Sufficient part of) Theorem B

$$\Downarrow$$

(Sufficient part of) Theorem A

Ring

- **Associative**
- **Not necessarily commutative**
- **Not necessarily unital**

Semigroup

Non-empty set S equipped with an associative binary operation $S \times S \ni (s,t) \mapsto st \in S$.

Ring extension A/B

Means that $A \supseteq B$. The centralizer of B in A, denoted by $C_A(B)$, is the set of elements in A that commute with every element of B.
If $C_A(B) = B$, then B is said to be a maximal commutative subring of A. The set $C_A(A)$ is called the center of A and is denoted by $Z(A)$.

Ideal intersection property

The ring extension A/B is said to have the ideal intersection property if every non-zero ideal of A has non-zero intersection with B.

Weak degree map

(Nystedt and Öinert 2014)

Let A/B be a ring extension. Suppose that $d : A \to \mathbb{Z}_{\geq 0}$ is a function and let I be an ideal of A. We say that $a \in I$ is I-minimal if

$$d(a) = \min\{d(x) \mid x \in I \text{ and } d(x) > 0\}.$$

We say that d is a weak degree map for A/B if it satisfies the following two conditions:

(d1) If $a \in A$, then $d(a) = 0$ if and only if $a = 0$.

(d2) For every non-zero ideal I of A there is an I-minimal element a such that for all $b \in B$ the inequality $d(ab - ba) < d(a)$ holds.

Proposition

If A/B has a weak degree map, then $A/C_A(B)$ has the ideal intersection property.

Proof

Suppose that $d : A \to \mathbb{Z}_{\geq 0}$ is a weak degree map for A/B. Let I be a non-zero ideal of A. From (d2) it follows that there is an I-minimal element a such that for all $b \in B$ the inequality $d(ab - ba) < d(a)$ holds. From (d1) and the definition of I-minimality it follows that for all $b \in B$ the relation $ab - ba = 0$ holds. Thus $a \in C_A(B) \cap I$.

System

Let R be a ring and let S be a semigroup. The ring R is called a system if there to every $s \in S$ is an additive subgroup R_s of R such that

- $R = \sum_{s \in S} R_s$ **and**
- $R_s R_t \subseteq R_{st}$ **for** $s, t \in S$**.**

If R is a system, then it is called strong if for all $s, t \in S$ the equality $R_s R_t = R_{st}$ holds.

If R is a system, then it is called graded if $R = \oplus_{s \in S} R_s$.

If R is graded, then it is called strongly graded if it is also a strong system.

Idempotent coherent

Let R be a system. Let $E(S)$ denote the set of idempotents of S and put $R_0 = \sum_{e \in E(S)} R_e$. We say that R is idempotent coherent if for all $s \in S$ the inclusions $R_0 R_s \subseteq R_s$ and $R_s R_0 \subseteq R_s$ hold. In that case, R_0 is a subring of R.

Weak degree map

Let R be a system. Define a function $d : R \to \mathbb{Z}_{\geq 0}$ in the following way.

If $r = 0$, then put $d(r) = 0$.

If $r \neq 0$, then there is $n \in \mathbb{N}$, $s_1, \ldots, s_n \in S$ and non-zero $r_i \in R_{s_i}$, for $i = 1, \ldots, n$, such that $r = \sum_{i=1}^{n} r_i$. Amongst all such representations of r, choose one with n minimal. Put $d(r) = n$.

Minimally non-degenerate

Let R be a system. We say that R is left (right) minimally non-degenerate if for all non-zero ideals I of R and all I-minimal elements r with $d(r) = n$ and all $r_i \in R_{s_i}$, for $i = 1, \ldots, n$, such that $r = \sum_{i=1}^{n} r_i$, there is $t \in S$ and $i \in \{1, \ldots, n\}$ such that $ts_i \in E(S)$ ($s_i t \in E(S)$) and $R_t r$ is non-zero ($r R_t$ is non-zero).

Proposition (*)

Let R be a system. If R is idempotent coherent and left (right) minimally non-degenerate, then $R/C_R(Z(R_0))$ has the ideal intersection property.

Show that d is a weak degree map for $R/Z(R_0)$.

System ideal

Let R be a system. Let I be an ideal of R. We say that I is a system ideal if $I = \sum_{s \in S} I \cap R_s$. We say that R is system simple if R and $\{0\}$ are the only system ideals of R. Note that if R is simple, then R is system simple.

Proof of Theorem D

Let I be a non-zero ideal of R. From Prop. (*) it follows that the additive group $J = I \cap C_R(Z(R_0))$ is non-zero. From the assumption $C_R(Z(R_0)) \subseteq R_0$ it follows that $J \subseteq R_0$. Thus $K = RJR + J$ is a non-zero system ideal of R. From system simplicity of R it follows that $K = R$. Thus $R = K = RJR + J \subseteq RIR + I = I$ and hence $R = I$.

Modules

Let A be a ring and let M be a left A-module.

We say that M is...

...unital if there exists $a \in A$ such that for all $m \in M$ the equality $am = m$ holds;

...s-unital if for all $m \in M$ there exists $a \in A$ such that $am = m$;

...unitary if $AM = M$.

Inverse semigroup

Let S be a semigroup. S is said to be an inverse semigroup if there for all $s \in S$ exists a unique $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$.

Partial order

Let S be an inverse semigroup. Define a partial order $\leq$ on S by saying that if $s, t \in S$, then $s \leq t$ if $s = ts^*s$ ($\Leftrightarrow$ $s = ss^*t$).

Remark

Let S be an inverse semigroup and let R be a system. Take $s \in S$. Then:

- $R_s R_{s^*}$ **is a ring**
- R_s **is an** $R_s R_{s^*}$**-**$R_{s^*} R_s$**-bimodule.**

Epsilon-strong system

Let S be an inverse semigroup and let R be a system. Let X denote either of:

- **”unital”**
- **”s-unital”**
- **”unitary”.**

We say that R is left (right) X epsilon-strong if for all $s \in S$ the left $R_s R_{s^*}$-module (right $R_{s^*} R_s$-module) R_s is X. If R is both left and right X epsilon-strong, then we say that R is X epsilon-strong.

Remark

Let S be an inverse semigroup and let R be a system. The following are equivalent:

- **R is a left unitary epsilon-strong system**
- **R is a right unitary epsilon-strong system**
- **R is a unitary epsilon-strong system**
- **R is symmetric, that is for all $s \in S$:**

$$R_s R_{s^*} R_s = R_s.$$

Proposition

If R is a coherent and left (right) s-unital epsilon-strong system, then R is left (right) minimally non-degenerate.

Using this proposition and Theorem D we can therefore deduce Theorem C.

Using Theorem C we can prove:

Theorem B$'$

Suppose that π is an s-unital partial action of an inverse semigroup S on an associative (but not necessarily commutative) s-unital ring A. If A is S-simple and $C_{A \rtimes_\pi S}(Z(A)) \subseteq A$, then the skew inverse semigroup ring $A \rtimes_\pi S$ is simple.

Using Theorem B$'$ we can prove:

Theorem A$'$

Suppose that K is a simple and associative (but not necessarily commutative or unital) ring with the property that $Z(K)$ contains a set of s-units for K. If G is a Hausdorff, ample, effective and minimal groupoid, then the Steinberg algebra $A_K(G)$ is simple.

Non-Hausdorff Theorem A

(Clark, Exel, Pardo, Sims, Starling 12/6-2018)

If K is a field and G is a second-countable, ample groupoid such that G_0 is Hausdorff, then $A_K(G)$ is simple if and only if the following three conditions are satisfied:

(1) G is minimal

(2) G is effective

(3) for all non-zero $f \in A_K(G)$ the support of f has non-empty interior.

Remark

There do not seem to be a way to prove Non-Hausdorff Theorem A using Theorem B.

Question

Is it possible to prove Non-Hausorff Theorem A using Theorem D (or even Theorem C) ?

Thank you!