Classification of Low Dimensional Hom-Lie Algebras

Elvice ONGONGA

Mälardalen University, Sweden

SNAG Workshop

September 26, 2018

Overview

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3. $4-$Dimensional Hom-Lie algebras
4. Nilpotent linear endomorphism

Introduction

Definition

[2] A Hom-Lie algebra is a triple $(V, [\cdot,\cdot], \alpha)$ consisting of a linear space V, bilinear map $[\cdot,\cdot] : V \times V \longrightarrow V$ and a linear space homomorphism $\alpha : V \to V$ satisfying

$$[x, y] = -[y, x] \qquad (skew - symettry) \tag{1}$$

$$\circlearrowleft_{x,y,z} [\alpha(x), [y, z]] = 0 \qquad (Hom - Jacobi\ identity) \tag{2}$$

for all $x, y, z \in V$, where $\circlearrowleft_{x,y,z}$ denotes summation over the cyclic permutation on x, y, z.

Equations of structure constants

Let $\{e_1, e_2, \ldots, e_n\}$ be basis for V.

The structure constant equations are given by;

$$[e_i, e_j] = \sum_{s=1}^{n} C_{ij}^{s} e_s \tag{3}$$

and

$$\alpha(e_i) = \sum_{t=1}^{n} a_{it} e_t \tag{4}$$

Replacing 3 and 4 in the Hom-Jacobi identity, we have a system of polynomial equations;

$$\sum_{s,t=1}^{n} a_{it} C_{jk}^{s} C_{ts}^{r} + a_{jt} C_{ki}^{s} C_{ts}^{r} + a_{kt} C_{ij}^{s} C_{ts}^{r} = 0, \qquad r = 1, 2, \ldots, n. \tag{5}$$

Replacing 3 and 4 in the Hom-Jacobi identity, we have a system of polynomial equations;

$$\sum_{s,t=1}^{n} a_{it} C_{jk}^{s} C_{ts}^{r} + a_{jt} C_{ki}^{s} C_{ts}^{r} + a_{kt} C_{ij}^{s} C_{ts}^{r} = 0, \qquad r = 1, 2, \ldots, n. \qquad (5)$$

Writing the equations as linear in $a'_{it}s$, we rewrite 5 as;

$$\sum_{t=1}^{n} \left\{ a_{it} \left(\sum_{s=1}^{n} C_{jk}^{s} C_{ts}^{r} \right) + a_{jt} \left(\sum_{s=1}^{n} C_{ki}^{s} C_{ts}^{r} \right) + a_{kt} \left(\sum_{s=1}^{n} C_{ij}^{s} C_{ts}^{r} \right) \right\} = 0, \quad (6)$$

$1 \leq i < j < k \leq n, \; r = 1, 2, \ldots, n.$

Thus 6 becomes

$$Ma_\alpha = 0$$

where column matrix a_α is;

$$a_\alpha = \begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \\ a_{12} \\ \vdots \\ a_{n2} \\ \vdots \\ \vdots \\ a_{nn} \end{pmatrix}$$

M is a matrix with n^2 columns and $\binom{n}{3} \cdot n$ rows.

3−Dimensional Hom Lie algebras

In three dimension Hom-Lie algebras the system of polynomial equations is obtained from 5. Together with the skew- symmetry condition, the equation becomes;

$$\sum_{\substack{m,n=1 \\ m<n}}^{3} \left(\circlearrowleft_{i,j,k} \left(a_{im} C^{n}_{jk} C^{r}_{mn} - a_{in} C^{m}_{jk} C^{r}_{mn} \right) \right) = 0 \qquad (7)$$

for $r = 1, 2, 3$ and $1 \leq i < j < k \leq 3$, where the symbol $\circlearrowleft_{i,j,k}$ denotes a summation over the cyclic permutation on i, j, k.

Matrix M is a 3 by 9 matrix given as;

$$M = \begin{pmatrix} M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} & M_{1,5} & M_{1,6} & M_{1,7} & M_{1,8} & M_{1,9} \\ M_{2,1} & M_{2,2} & M_{2,3} & M_{2,4} & M_{2,5} & M_{2,6} & M_{2,7} & M_{2,8} & M_{2,9} \\ M_{3,1} & M_{3,2} & M_{3,3} & M_{3,4} & M_{3,5} & M_{3,6} & M_{3,7} & M_{3,8} & M_{3,9} \end{pmatrix}$$

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$$M_{r,1} = (C_{23}^2 C_{12}^r + C_{23}^3 C_{13}^r) \quad M_{r,2} = (-C_{13}^2 C_{12}^r - C_{13}^3 C_{13}^r)$$
$$M_{r,3} = (C_{12}^2 C_{12}^r + C_{12}^3 C_{13}^r) \quad M_{r,4} = (C_{23}^3 C_{23}^r - C_{23}^1 C_{12}^r)$$
$$M_{r,5} = (C_{13}^1 C_{12}^r - C_{13}^3 C_{23}^r) \quad M_{r,6} = (C_{12}^3 C_{23}^r - C_{12}^1 C_{12}^r)$$
$$M_{r,7} = (-C_{23}^1 C_{13}^r - C_{23}^2 C_{23}^r) \quad M_{r,8} = (C_{13}^1 C_{13}^r + C_{13}^2 C_{23}^r)$$
$$M_{r,9} = (-C_{12}^1 C_{13}^r - C_{12}^2 C_{23}^r)$$

for $r = 1, 2, 3.$

M represents a linear map $L : \mathbb{K}^9 \to \mathbb{K}^3$, and hence a_α must be in the $\ker(L)$ for us to realize a Hom-Lie algebra.

From nullity-rank theorem,

Thus Rank $(M) = 3 \iff \dim\ (\ker(L) = 6$.

Let us denote the determinant of any three columns i, j, k as (i, j, k) and let C the matrix of structure constants given as

$$C = \begin{pmatrix} C_{12}^1 & C_{12}^2 & C_{12}^3 \\ C_{13}^1 & C_{13}^2 & C_{13}^3 \\ C_{23}^1 & C_{23}^2 & C_{23}^3 \end{pmatrix}$$

Proposition

Let M represent a linear map $L : \mathbb{K}^9 \to \mathbb{K}^3$ and M be the matrix of coefficients of the bilinear map $[\cdot, \cdot]$. A Hom-Lie algebra structure is achieved if a_α, the column vector involving coefficients of the endomorphism map α, is in $\ker(L)$, and $\ker(L)$ attains minimum dimension 6, if and only if $\det C$ is non-zero.

Proof.

For Rank $(M) = 3$, then there exists a non-zero sub-determinant of a 3×3 sub-matrix of M.

From the computations all the non-zero sub-determinants have $\det C$ as a factor.

$\square$

For example;

$$(1,2,4) = C^3_{23}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$(1,2,5) = -C^3_{13}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$(1,2,6) = C^3_{12}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$(1,2,7) = -C^2_{23}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$(1,2,8) = C^2_{13}(C^2_{13}C^3_{23} - C^3_{13}C^2_{23})(\det C)$$

$$\vdots$$

4−Dimensional Hom-Lie algebras

From 5, we have the following system of polynomial equations;

$$\sum_{\substack{m,n=1 \\ m<n}}^{4} \left(\circlearrowleft_{i,j,k} \left(a_{im} C_{jk}^{n} C_{mn}^{r} - a_{1n} C_{jk}^{m} C_{mn}^{r}\right)\right) = 0 \quad \text{for } r = 1, 2, 3, 4, \tag{8}$$

where the symbol $\circlearrowleft_{i,j,k}$ denotes a summation over the cyclic permutation on i, j, k with $i, j, k = 1, 2, 3, 4$ and $i < j < k$.

Matrix M is a square matrix of order 16×16 and hence we can only realize Hom-Lie algebras when $\det M = 0$.

The matrix of order 16×16 involves complicated computations and so we can proceed to reduce the endomorphism α into its reduced basis form,

$$\begin{pmatrix} a_{11} & a_{12} & 0 & 0 \\ 0 & a_{22} & a_{23} & 0 \\ 0 & 0 & a_{33} & a_{34} \\ 0 & 0 & 0 & a_{44} \end{pmatrix}$$

Nilpotent linear endomorphism 3-Dimension

In three dimension case, the linear endomorphism α is given in its reduced basis form as;

$$\alpha = \begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

We consider the Jordan forms of α and proceed to find all the non-isomorphic families of Hom-Lie algebras given by each of the following cases.

$$\alpha_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \alpha_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \alpha_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Case 1

We get the following corresponding Hom-Lie algebras

$\mathcal{H}_{\mathbf{1}}^{\mathbf{3}}$

$$[e_1, e_2] = C_{12}^1\ e_1 + C_{12}^2\ e_2 + C_{12}^3\ e_3$$
$$[e_1, e_3] = C_{13}^1\ e_1 + C_{13}^2\ e_2 + C_{13}^3\ e_3$$
$$[e_2, e_3] = C_{23}^2\ e_2$$

$\mathcal{H}_{\mathbf{2}}^{\mathbf{3}}$

$$[e_1, e_2] = C_{23}^3\ e_1 + \frac{C_{23}^2\ C_{23}^3}{C_{23}^1}\ e_2 + \frac{(C_{23}^3)^2}{C_{23}^1}\ e_3$$
$$[e_1, e_3] = C_{13}^1\ e_1 + C_{13}^2\ e_2 + C_{13}^3\ e_3$$
$$[e_2, e_3] = C_{23}^1\ e_1 + C_{23}^2\ e_2 + C_{23}^3\ e_3$$

Case 2

We get the following corresponding Hom-Lie algebras

$\mathcal{H}_{\mathbf{3}}^{\mathbf{3}}$

$$[e_1, e_2] = C_{12}^1\ e_1 + C_{12}^2\ e_2 + C_{12}^3\ e_3$$
$$[e_1, e_3] = C_{13}^3\ e_3$$
$$[e_2, e_3] = C_{23}^1\ e_1 + C_{23}^2\ e_2 + C_{23}^3\ e_3$$

$\mathcal{H}_{\mathbf{4}}^{\mathbf{3}}$

$$[e_1, e_2] = C_{12}^1\ e_1 + C_{12}^2\ e_2 + C_{12}^3\ e_3$$
$$[e_1, e_3] = C_{13}^1\ e_1 + C_{13}^2\ e_2 + C_{13}^3\ e_3$$
$$[e_2, e_3] = -\frac{(C_{13}^1)^2}{C_{13}^2}\ e_1 - C_{13}^1\ e_2 - \frac{C_{13}^1\ C_{13}^3}{C_{13}^2}\ e_3$$

Case 3

We get the following corresponding Hom-Lie algebras

$\mathcal{H}_{\mathbf{5}}^{\mathbf{3}}$

$$[e_1, e_2] = C_{12}^1\ e_1 + C_{12}^2\ e_2 + C_{12}^3\ e_3$$
$$[e_1, e_3] = -C_{23}^3\ e_2 + C_{13}^3\ e_3$$
$$[e_2, e_3] = C_{23}^2\ e_2 + C_{23}^3\ e_3$$

$\mathcal{H}_{\mathbf{6}}^{\mathbf{3}}$

$$[e_1, e_2] = C_{12}^1\ e_1 + C_{12}^2\ e_2 + C_{12}^3\ e_3$$
$$[e_1, e_3] = C_{13}^2\ e_2 + C_{13}^3\ e_3$$
$$[e_2, e_3] = 0$$

Case 3

$\mathcal{H}_{\mathbf{7}}^{\mathbf{3}}$

$$[e_1, e_2] = \frac{(C_{13}^1)^2 + C_{13}^2\ C_{23}^1 + C_{23}^1\ C_{23}^3}{C_{23}^1}\ e_1 + \frac{C_{13}^1\ C_{13}^2 + C_{13}^2\ C_{23}^2 + C_{23}^2\ C_{23}^3}{C_{23}^1}\ e_2 + \frac{C_{13}^1\ C_{13}^3 + C_{13}^2\ C_{23}^3 + (C_{23}^2)^2}{C_{23}^1}\ e_3$$

$$[e_1, e_3] = C_{13}^1\ e_1 + C_{13}^2\ e_2 + C_{13}^3\ e_3$$

$$[e_2, e_3] = C_{23}^1\ e_1 + C_{23}^2\ e_2 + C_{23}^3\ e_3$$

Isomorphisms

Definition

An isomorphism of Hom-Lie algebras $\phi : (\mathcal{L}_1, \alpha_1) \longrightarrow (\mathcal{L}_1, \alpha_2)$ is an algebra isomorphism from $\mathcal{L}_1$ to $\mathcal{L}_2$ such that $\phi \circ \alpha_1 = \alpha_2 \circ \phi$. That is, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{L}_1 & \xrightarrow{\phi} & \mathcal{L}_2 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} \\ \mathcal{L}_1 & \xrightarrow{\phi} & \mathcal{L}_2 \end{array}$$

If we define $\phi(e_i) = \sum_{k=1}^{3} c_{ik} f_k$,

$$\mathcal{H}_1^3 \cong \mathcal{H}_3^3 \ \text{ and } \mathcal{H}_2^3 \cong \mathcal{H}_4^3$$

the isomorphism ϕ given by;

$$\phi(e_1) = f_2 \quad \phi(e_2) = f_3 \quad \phi(e_3) = f_1$$

Result

Proposition

Any three dimensional Hom-Lie algebra $(\mathcal{H}, [\cdot,\cdot], \alpha)$ *with* α *nilpotent is isomorphic to some Hom-Lie algebras from the following family of Hom-Lie algebras*

$$(\mathcal{H}_1^3, \alpha_1), \quad (\mathcal{H}_2^3, \alpha_1), \quad (\mathcal{H}_5^3, \alpha_3), \quad (\mathcal{H}_6^3, \alpha_3), \quad (\mathcal{H}_7^3, \alpha_3)$$

Work in Progress

Finding all the non-isomorphic representatives from the families of Hom-Lie algebras.

So far:

$$\mathcal{H}_1^3 \ncong \mathcal{H}_2^3$$

$$\mathcal{H}_5^3 \ncong \mathcal{H}_7^3$$

$$\mathcal{H}_6^3 \ncong \mathcal{H}_7^3$$

$$(\mathcal{H}_i^3, \alpha_1) \ncong (\mathcal{H}_i^3, \alpha_3)$$

Nilpotent linear endomorphism 4-Dimension

We consider the Jordan forms of α and proceed to find all the non-isomorphic families of Hom-Lie algebras given by each of the following cases.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Case 1

This is work in progress in order to classify all the Hom-Lie algebras fully.

Example

$\mathcal{H}_{\mathbf{1}}^{\mathbf{4}}$

$$[e_1, e_2] = C_{12}^2 \; e_2$$
$$[e_1, e_3] = C_{13}^1 \; e_1 + C_{13}^2 \; e_2 + C_{13}^3 \; e_3 + C_{13}^4 \; e_4$$
$$[e_1, e_4] = C_{14}^1 \; e_1 + C_{14}^2 \; e_2 + C_{14}^3 \; e_3 + C_{14}^4 \; e_4$$
$$[e_2, e_3] = C_{23}^1 \; e_1 + C_{23}^2 \; e_2 + C_{23}^4 \; e_4$$
$$[e_2, e_4] = \frac{C_{12}^2 \; C_{23}^1}{C_{23}^4} \; e_2$$
$$[e_3, e_4] = C_{34}^2 \; e_2$$

Case 1

Example

$\mathcal{H}_{\mathbf{2}}^{\mathbf{4}}$

$$[e_1, e_2] = C_{12}^2\ e_2$$
$$[e_1, e_3] = C_{13}^1\ e_1 + C_{13}^2\ e_2 + C_{13}^3\ e_3 + C_{13}^4\ e_4$$
$$[e_1, e_4] = C_{14}^1\ e_1 + C_{14}^2\ e_2 + C_{14}^3\ e_3 + C_{14}^4\ e_4$$
$$[e_2, e_3] = \frac{C_{12}^2\ C_{34}^1}{C_{34}^3}\ e_2$$
$$[e_2, e_4] = C_{24}^2\ e_2$$
$$[e_3, e_4] = C_{34}^1\ e_1 + C_{34}^2\ e_2 + C_{34}^3\ e_3$$

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Thank you!