Chain conditions on epsilon-strongly graded rings

with applications to Leavitt path algebras

Daniel Lännström

Blekinge Institute of Technology
PhD Student. Supervisors: Johan Öinert (BTH), Stefan Wagner (BTH), Patrik Nystedt (HV)

The 1st SNAG meeting 2018, Linköping

Outline

Part 1

General theorems about noetherian and artinian (unital, associative) epsilon-strongly graded rings.

Outline

Part 1

General theorems about noetherian and artinian (unital, associative) epsilon-strongly graded rings.

Main application:

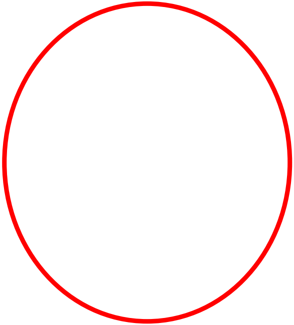
Part 2

Characterization of noetherian and artinian Leavitt path algebras with coefficients in a unital ring

Reference

Daniel Lännström. Chain conditions for epsilon-strongly graded rings with applications to Leavitt path algebras. Preprint. arXiv:1808.10163, 2018.

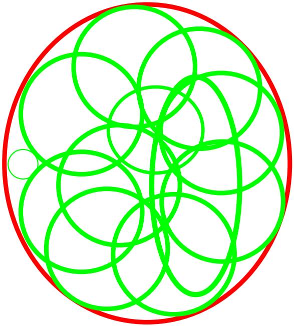
Overview



Legend

Ring S (unital, associative)

Overview

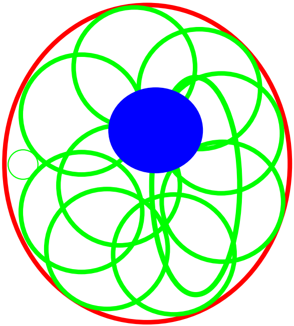


Legend

Ring S

G-grading

Overview



Legend

Ring S

G-grading

Principal component $S_e \subseteq S$

Overview

Idea 1

("Hilbert Basis Theorem") Under certain conditions on the grading, the ring S is noetherian/artinian iff S_e is noetherian/artinian.

Idea 2 (application)

The principal component S_e is often easier to understand than S. This is especially true for Leavitt path algebras.

Part 1: "Hilbert basis theorem" for epsilon-strongly graded rings

Definition

Let G be a group and let S be a ring.

Group graded rings

Definition

Let G be a group and let S be a ring.
A *G-grading* of S is a decomposition,

$$S = \bigoplus_{g \in G} S_g, \tag{1}$$

such that for all $g, h \in G$,

$$S_g S_h \subseteq S_{gh}. \tag{2}$$

Group graded rings

Definition

Let G be a group and let S be a ring.
A *G-grading* of S is a decomposition,

$$S = \bigoplus_{g \in G} S_g, \tag{1}$$

such that for all $g, h \in G$,

$$S_g S_h \subseteq S_{gh}. \tag{2}$$

If $S_g S_h = S_{gh}$ it is a *strong G-grading*.

Examples

Example

The group ring $R[G] = \bigoplus_{g \in G} R\delta_g$ where the δ_g's are formal symbols. Multiplication is defined by the rule:

$$(r_1\delta_g)(r_2\delta_h) = r_1 r_2 \delta_{gh}. \tag{3}$$

Examples

Example

The group ring $R[G] = \bigoplus_{g \in G} R\delta_g$ where the δ_g's are formal symbols. Multiplication is defined by the rule:

$$(r_1\delta_g)(r_2\delta_h) = r_1 r_2 \delta_{gh}. \tag{3}$$

Putting,

$$S_g := R\delta_g \tag{4}$$

gives strong G-gradation.

Examples

Example

The ring of Laurent polynomial ring $K[X, X^{-1}]$ can be graded as follows:

Examples

Example

The ring of Laurent polynomial ring $K[X, X^{-1}]$ can be graded as follows:

$$K[X, X^{-1}] = \bigoplus_{i \in \mathbb{Z}} KX^i. \tag{5}$$

Strong $\mathbb{Z}$-grading.

"Hilbert Basis Theorem"

Theorem

(A. Bell (1987) [1]) Let G be a polycyclic-by-finite group and let S be strongly G-graded. Then S is left (right) noetherian if and only if S_e is left (right) noetherian.

Epsilon-strongly graded rings

Generalization of strongly graded rings.

Definition

(Nystedt, Öinert, Pinedo [2]) A G-grading,

$$S = \bigoplus_{g \in G} S_g . \tag{6}$$

is called *epsilon-strong* if, for all $g \in G$,

Epsilon-strongly graded rings

Generalization of strongly graded rings.

Definition

(Nystedt, Öinert, Pinedo [2]) A G-grading,

$$S = \bigoplus_{g \in G} S_g. \tag{6}$$

is called *epsilon-strong* if, for all $g \in G$,

1. $S_g S_{g^{-1}} S_g = S_g$ (symmetric)

Epsilon-strongly graded rings

Generalization of strongly graded rings.

Definition

(Nystedt, Öinert, Pinedo [2]) A G-grading,

$$S = \bigoplus_{g \in G} S_g. \tag{6}$$

is called *epsilon-strong* if, for all $g \in G$,

1. $S_g S_{g^{-1}} S_g = S_g$ (symmetric)
2. $S_g S_{g^{-1}} \subseteq S_e$ is a unital S_e-ideal.

Example 2

Example

(J. Öinert) Consider $M_2(\mathbb{C})$ with the following $\mathbb{Z}$-grading.

Example 2

Example

(J. Öinert) Consider $M_2(\mathbb{C})$ with the following $\mathbb{Z}$-grading.

$$S_0 = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}, S_1 = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}, S_{-1} = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}$$

Example 2

Example

(J. Öinert) Consider $M_2(\mathbb{C})$ with the following $\mathbb{Z}$-grading.

$S_0 = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}, S_1 = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}, S_{-1} = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}$

$S_n = \{0\}$ for $|n| > 1$.

Example 2

Example

(J. Öinert) Consider $M_2(\mathbb{C})$ with the following $\mathbb{Z}$-grading.

$S_0 = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}, S_1 = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}, S_{-1} = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}$

$S_n = \{0\}$ for $|n| > 1$.

Check grading. Check symmetric.

Example 2

Example

(J. Öinert) Consider $M_2(\mathbb{C})$ with the following $\mathbb{Z}$-grading.

$S_0 = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}, S_1 = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}, S_{-1} = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}$

$S_n = \{0\}$ for $|n| > 1$.

Check grading. Check symmetric.

$S_1 S_{-1} = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix} \subsetneq S_0$

Example 2

Example

(J. Öinert) Consider $M_2(\mathbb{C})$ with the following $\mathbb{Z}$-grading.

$S_0 = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}, S_1 = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}, S_{-1} = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}$

$S_n = \{0\}$ for $|n| > 1$.

Check grading. Check symmetric.

$S_1 S_{-1} = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix} \subsetneq S_0$

Multiplicative identity $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Example 2

Example

(J. Öinert) Consider $M_2(\mathbb{C})$ with the following $\mathbb{Z}$-grading.

$S_0 = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & \mathbb{C} \end{pmatrix}, S_1 = \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}, S_{-1} = \begin{pmatrix} 0 & 0 \\ \mathbb{C} & 0 \end{pmatrix}$

$S_n = \{0\}$ for $|n| > 1$.

Check grading. Check symmetric.

$S_1 S_{-1} = \begin{pmatrix} \mathbb{C} & 0 \\ 0 & 0 \end{pmatrix} \subsetneq S_0$

Multiplicative identity $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Similarly, $S_{-1} S_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{C} \end{pmatrix}$.

Subclasses of epsilon-strongly graded rings

1. Strongly-graded rings

Subclasses of epsilon-strongly graded rings

1. Strongly-graded rings
2. Unital partial crossed products

Subclasses of epsilon-strongly graded rings

1. Strongly-graded rings
2. Unital partial crossed products
3. Leavitt path algebras

Subclasses of epsilon-strongly graded rings

1. Strongly-graded rings
2. Unital partial crossed products
3. Leavitt path algebras

Theorem

(Nystedt, Öinert [3]) Let R be a ring and let E be a finite directed graph. Then $L_R(E)$ is canonically epsilon-strongly $\mathbb{Z}$-graded.

"Hilbert Basis Theorem for epsilon-strongly graded rings"

Theorem

(Lännström, 2018) Let G be a polycyclic-by-finite group and let S be an epsilon-strongly G-graded ring. Then, S is left/right noetherian if and only if S_e is left/right noetherian.

Theorem

(Lännström, 2018) Let G be a torsion-free group and let S be an epsilon-strongly G-graded ring. Then, S is left/right artinian if and only if S_e is left/right artinian and $S_g \neq \{0\}$ for finitely many $g \in G$.

Remarks

Remark

Polycyclic-by-finite is the largest known class of group such that the group ring $\mathbb{C}[G]$ is one-sided noetherian.

Remarks

Remark

Polycyclic-by-finite is the largest known class of group such that the group ring $\mathbb{C}[G]$ is one-sided noetherian.

Remark

Passman gave an example of an artinian twisted group ring (over a field) by an infinte p-group.

Remarks

Remark

Polycyclic-by-finite is the largest known class of group such that the group ring $\mathbb{C}[G]$ is one-sided noetherian.

Remark

Passman gave an example of an artinian twisted group ring (over a field) by an infinte p-group.

Part 2: Applications

Applications

Application 1

Characterizations of noetherian and artinian unital partial crossed products. Generalizes previous work on partial skew group rings by Carvalho, Cortes, Ferrero.

Applications

Application 1

Characterizations of noetherian and artinian unital partial crossed products. Generalizes previous work on partial skew group rings by Carvalho, Cortes, Ferrero.

Application to Leavitt path algebras.

Key point

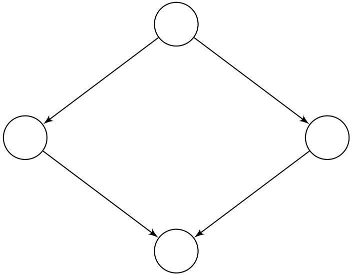
$\mathbb{Z}$ is both polycyclic-by-finite and torsion-free. Hence, we can apply the above theorems to the special case of Leavitt path algebras (which are epsilon-strongly $\mathbb{Z}$-graded)!!

Leavitt path algebras

Algebraic analogues of graph C^*-algebra and a generalization of Leavitt algebras. (G. Abrams, G. Aranda Pino, P. Ara, M. A. Moreno, E. Pardo).

Algebraic analogues of graph C^*-algebra and a generalization of Leavitt algebras. (G. Abrams, G. Aranda Pino, P. Ara, M. A. Moreno, E. Pardo).

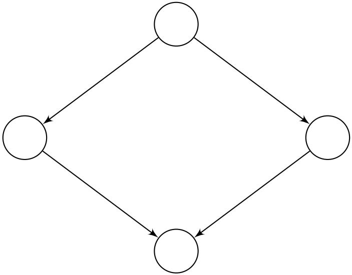
Let R be a ring and let E be a directed graph.



Leavitt path algebras

Algebraic analogues of graph C^*-algebra and a generalization of Leavitt algebras. (G. Abrams, G. Aranda Pino, P. Ara, M. A. Moreno, E. Pardo).

Let R be a ring and let E be a directed graph.

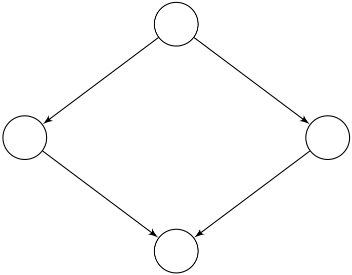


Attach an R-algebra $L_R(E)$ to the graph E.

Leavitt path algebras

Algebraic analogues of graph C^*-algebra and a generalization of Leavitt algebras. (G. Abrams, G. Aranda Pino, P. Ara, M. A. Moreno, E. Pardo).

Let R be a ring and let E be a directed graph.



Attach an R-algebra $L_R(E)$ to the graph E.

LPA definition

Definition

Let R be a ring and $E = (E^0, E^1, s, r)$ be a directed graph. The *Leavitt path algebra* attached to E with coefficients in R is the R-algebra generated by the symbols:

1. $\{v \mid v \in E^0\}$,
2. $\{f \mid f \in E^1\}$,
3. $\{f^* \mid f \in E^1\}$.

...

LPA definition (cont)

Definition

...

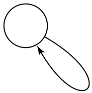
subject to the following relations:

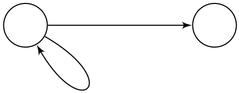
1. $v_i v_j = \delta_{i,j} v_i$ for all $v_i, v_j \in E^0$,
2. $s(f)f = fr(f) = f$ and $r(f)f^* = f^* s(f) = f^*$ for all $f \in E^1$,
3. $f^* f' = \delta_{f,f'} r(f)$ for all $f, f' \in E^1$,
4. $\sum_{f \in E^1, s(f)=v} ff^* = v$ for all $v \in E^0$ for which $s^{-1}(v)$ is non-empty and finite.

Finiteness condition of graphs

Definition

We say that E satisfies Condition (NE) if there exists no cycle with an exit.





LPA example 1

Let E be the following graph:

LPA example 1

Let E be the following graph:

acyclic graph

LPA example 1

Let E be the following graph:

acyclic graph

$$L_R(E) \cong R \quad (7)$$

LPA example 1

Let E be the following graph:

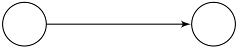
acyclic graph

$$L_R(E) \cong R \tag{7}$$

Noetherian/artinian iff R is noetherian/artinian.

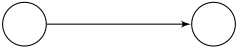
LPA example 2

Let E be the following graph:



LPA example 2

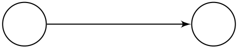
Let E be the following graph:



acyclic graph

LPA example 2

Let E be the following graph:

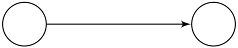


acyclic graph

$$L_R(E) \cong M_2(R) \tag{8}$$

LPA example 2

Let E be the following graph:



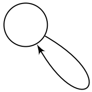
acyclic graph

$$L_R(E) \cong M_2(R) \tag{8}$$

Semisimple ring iff R is a division ring.

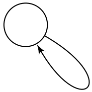
LPA example 3

Let E be the following graph:



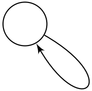
LPA example 3

Let E be the following graph:



Condition (NE)

Let E be the following graph:

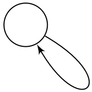


Condition (NE)

$$L_R(E) \cong R[X, X^{-1}] \tag{9}$$

LPA example 3

Let E be the following graph:



Condition (NE)

$$L_R(E) \cong R[X, X^{-1}] \tag{9}$$

R noetherian $\implies$ $R[X, X^{-1}]$ noetherian.

Applications to Leavitt path algebras

Generalized characterizations of noetherian and artinian Leavitt path algebras with coefficients in a unital ring R.

Applications to Leavitt path algebras

Generalized characterizations of noetherian and artinian Leavitt path algebras with coefficients in a unital ring R.

For fields K: (Abrams, Ara, Siles Molina)
Conditions on $E \iff$ conditions on $L_K(E)$

Applications to Leavitt path algebras

Generalized characterizations of noetherian and artinian Leavitt path algebras with coefficients in a unital ring R.

For fields K: (Abrams, Ara, Siles Molina)
Conditions on $E \iff$ conditions on $L_K(E)$
Ad-hoc methods

Generalized characterizations of noetherian and artinian Leavitt path algebras with coefficients in a unital ring R.

For fields K: (Abrams, Ara, Siles Molina)
Conditions on $E \iff$ conditions on $L_K(E)$
Ad-hoc methods

For commutative ring R: (Steinberg [4])

Generalized characterizations of noetherian and artinian Leavitt path algebras with coefficients in a unital ring R.

For fields K: (Abrams, Ara, Siles Molina)
Conditions on $E \iff$ conditions on $L_K(E)$
Ad-hoc methods

For commutative ring R: (Steinberg [4])
Conditions on E + conditions on $R \iff$ conditions on $L_R(E)$

Applications to Leavitt path algebras

Generalized characterizations of noetherian and artinian Leavitt path algebras with coefficients in a unital ring R.

For fields K: (Abrams, Ara, Siles Molina)
Conditions on $E \iff$ conditions on $L_K(E)$
Ad-hoc methods

For commutative ring R: (Steinberg [4])
Conditions on E + conditions on $R \iff$ conditions on $L_R(E)$
Utilizes framework of Steinberg algebras

Extension of (Steinberg 2018 [4]) using different techniques.

Applications to Leavitt path algebras

Extension of (Steinberg 2018 [4]) using different techniques.
Utilizes framework of epsilon-strongly graded rings

Extension of (Steinberg 2018 [4]) using different techniques.
Utilizes framework of epsilon-strongly graded rings

Theorem

(Lännström, 2018) Let R be a ring and E a directed graph. Then the following assertions hold.

Applications to Leavitt path algebras

Extension of (Steinberg 2018 [4]) using different techniques.
Utilizes framework of epsilon-strongly graded rings

Theorem

(Lännström, 2018) Let R be a ring and E a directed graph. Then the following assertions hold.

1. *$L_R(E)$ is left (right) noetherian if and only if R is left (right) noetherian and E is a finite graph containing no cycles with exits.*

Extension of (Steinberg 2018 [4]) using different techniques.
Utilizes framework of epsilon-strongly graded rings

Theorem

(Lännström, 2018) Let R be a ring and E a directed graph. Then the following assertions hold.

1. $L_R(E)$ *is left (right) noetherian if and only if* R *is left (right) noetherian and* E *is a finite graph containing no cycles with exits.*
2. $L_R(E)$ *is left (right) artinian if and only if* R *is left (right) artinian and* E *is a finite acyclic graph.*

Applications to Leavitt path algebras

Extension of (Steinberg 2018 [4]) using different techniques.
Utilizes framework of epsilon-strongly graded rings

Theorem

(Lännström, 2018) Let R be a ring and E a directed graph. Then the following assertions hold.

1. $L_R(E)$ *is left (right) noetherian if and only if* R *is left (right) noetherian and* E *is a finite graph containing no cycles with exits.*
2. $L_R(E)$ *is left (right) artinian if and only if* R *is left (right) artinian and* E *is a finite acyclic graph.*

Proof sketch

"Structure theorem for the principal component of LPAs"

Theorem

(cf. [5, Cor. 2.1.16]) Let E be a finite graph that satisfies Condition (NE). Then, there are integers positive integers $n_1, n_2, \ldots, n_k$ such that,

$$(L_R(E))_0 \cong M_{n_1}(R) \times M_{n_2}(R) \cdots \times M_{n_k}(R). \qquad (10)$$

Corollary: $(L_R(E))_0$ is Morita equivalent with R^k for some integer.

Corollary: $(L_R(E))_0$ left (right) noetherian/artininan iff R left (right) noetherian/artinian.

Proof sketch

Assume that R is left noetherian and E satisfies condition (NE).

Proof sketch

Assume that R is left noetherian and E satisfies condition (NE). By the previous corollary: $(L_R(E))_0$ is left noetherian.

Proof sketch

Assume that R is left noetherian and E satisfies condition (NE).
By the previous corollary: $(L_R(E))_0$ is left noetherian.
By "Hilbert basis theorem": $L_R(E)$ is left noetherian.

Semisimple LPAs

Not a full characterization. Straight-forward direction:

Semisimple LPAs

Not a full characterization. Straight-forward direction:

Theorem

Let R be a ring and let E be a directed graph. If $L_R(E)$ is semisimple then R is semisimple and E is acyclic.

Not a full characterization. Straight-forward direction:

Theorem

Let R be a ring and let E be a directed graph. If $L_R(E)$ is semisimple then R is semisimple and E is acyclic.

Partial converse!

Theorem

(Lännström, 2018) Let R be a ring and let E be a directed graph. If (i) R is semisimple and $n1_R$ is invertible for every integer $n \neq 0$, and (ii) E is acyclic, then $L_R(E)$ is semisimple.

Semisimple LPAs

Not a full characterization. Straight-forward direction:

Theorem

Let R be a ring and let E be a directed graph. If $L_R(E)$ is semisimple then R is semisimple and E is acyclic.

Partial converse!

Theorem

(Lännström, 2018) Let R be a ring and let E be a directed graph. If (i) R is semisimple and $n1_R$ is invertible for every integer $n \neq 0$, and (ii) E is acyclic, then $L_R(E)$ is semisimple.

"$n1_R$ is invertible for every integer $n \neq 0$" technical assumption not necessary condition!

Semisimple LPAs proof

Nystedt, Öinert and Pinedo [2] characterized when epsilon-strongly graded rings are separable over their principal component.

Semisimple LPAs proof

Nystedt, Öinert and Pinedo [2] characterized when epsilon-strongly graded rings are separable over their principal component.

Theorem

(Nystedt, Öinert, Pinedo [2]) Let S be an epsilon-strongly $\mathbb{Z}$-graded ring. Assume (i) that $\epsilon_i = 0$ for all but finitely many integers i and that (ii) $tr_\gamma(1)$ is invertible in S_0. If S_0 is semisimple, then S is semisimple.

A technical lemma:

Lemma

(Lännström, 2018) If E is a finite graph and R is a ring such that $n \cdot 1_R$ is invertible for each integer $n \neq 0$, then condition (ii) is satisfied.

Allen D Bell.
Localization and ideal theory in noetherian strongly group-graded rings.
Journal of Algebra, 105(1):76–115, 1987.

Patrik Nystedt, Johan Öinert, and Héctor Pinedo.
Epsilon-strongly graded rings, separability and semisimplicity.
Journal of Algebra, 514:1 – 24, 2018.

Patrik Nystedt and Johan Öinert.
Epsilon-strongly graded leavitt path algebras.
arXiv preprint arXiv:1703.10601, 2017.

Benjamin Steinberg.
Chain conditions on étale groupoid algebras with applications to leavitt path algebras and inverse semigroup algebras.
Journal of the Australian Mathematical Society, pages 1–9, 2018.

Gene Abrams, Pere Ara, and Mercedes Siles Molina.
Leavitt path algebras, volume 2191.
Springer, 2017.

Thank you for your attention!