KÄHLER-POISSON ALGEBRAS

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General ideas

- The following references [J. Arnlind and G. Huisken,2014] and [J. Arnlind, J Hoppe, and G. Huisken,2012] show that one may reformulate the Riemannian geometry of an embedded Kähler manifold M entirely in terms of the Poisson structure on the algebra smooth functions of M.
- We introduce Kähler-Poisson algebras as analogues of algebras of smooth functions on Kähler manifold.
- We prove that they share several properties with their classical counterparts on an algebraic level.
- We will see that the module of inner derivations of a Kähler-Poisson algebra is a finitely generated projective module, and allows for a unique metric and torsion-free connection whose curvature enjoys all the classical symmetries.

- Starting from a large class of Poisson algebras, we show that the algebras have an associated Kähler-Poisson algebras constructed as a localizations.
- We provide several examples in order to illustrate the novel concepts.

Section (1) KÄHLER-POISSON ALGEBRAS

We shall introduce a type of Poisson algebras, that resembles the smooth functions on an (isometrically) embedded almost Kähler manifold, in such a way that an analogue of Riemannian geometry may be developed. Our aim is to introduce Kähler-Poisson algebras below, and we will show that they are, in a natural way, metric Lie-Rinehart algebras, which implies that the results of Lie-Rinehart algebras can be applied,in particular, there exists a unique torsion-free metric connection on every Kähler-Poisson algebra.

Definition (1.1)

Let $\mathcal{A}$ be a poisson algebra over $\mathbb{K}$ and let $\{x^1,\ldots,x^m\}\subset\mathcal{A}$. Given $g_{ij}\in\mathcal{A}$, for $i,j=1,\ldots,m$, such that $g_{ij}=g_{ji}$ for $i,j=1,...,m$. We say that the triple $\mathcal{K}=(\mathcal{A},g,\{x^1,...,x^m\})$ is a **Kähler–Poisson algebra** if there exists $\eta\in\mathcal{A}$ such that

$$\sum_{i,j,k,l=1}^{m}\eta\{a,x^i\}g_{ij}\{x^j,x^k\}g_{kl}\{x^l,b\}=-\{a,b\} \tag{1}$$

for all $a,b\in\mathcal{A}$.

Remark. From now on, we shall use the differential geometric convention that repeated indices are summed over from 1 to m, and omit explicit summation symbols.

Given a Kähler–Poisson algebra $\mathcal{K}$, we let $\mathfrak{g}$ denote the $\mathcal{A}$-module generated by all inner derivations, i.e.

$$\mathfrak{g}=\{a_1\{c^1,\cdot\}+\cdots+a_N\{c^N,\cdot\}:a_i,c^i\in\mathcal{A}\text{ and }N\in\mathbb{N}\}.$$

It is a standard fact that $\mathfrak{g}$ is a Lie algebra over $\mathbb{K}$ with respect to

$$[\alpha, \beta](a) = \alpha(\beta(a)) - \beta(\alpha(a)).$$

The matrix $g = (g_{ij})$ induces a bilinear symmetric form on $\mathfrak{g}$, defined by

$$g(\alpha, \beta) = \alpha(x^i)g_{ij}\beta(x^j), \quad (2)$$

and we refer to g as the metric on $\mathfrak{g}$. To the metric g one may associate a map $\hat{g} : \mathfrak{g} \to \mathfrak{g}^*$ defined as

$$\hat{g}(\alpha)(\beta) = g(\alpha, \beta).$$

Proposition (1.2)

If $\mathcal{K} = (\mathcal{A}, \{x^1, ..., x^m\}, g)$ is a Kähler-Poisson algebra then the metric g is non-degenerate; i.e. the map $\hat{g} : \mathfrak{g} \to \mathfrak{g}^*$ is a module isomorphism.

Remark

It is easy to ckeck that $(\mathcal{A}, \mathfrak{g})$ is indeed a Lie-Rinehart algebra and, furthermore, Proposition (1.2) implies that $(\mathcal{A}, \mathfrak{g}, g)$ is a metric Lie-Rinehart algebra.

Let us now introduce some notation for Kähler-Poisson algebras. Thus, we set

$$\mathcal{D}^{ij} = \eta \mathcal{P}^i_k \mathcal{P}^{jk} = \eta \{x^i, x^l\} g_{lk} \{x^j, x^k\}$$
$$\mathcal{D}^i(a) = \eta \mathcal{P}^k(a) \mathcal{P}_k{}^i = \eta \{x^k, a\} g_{kl} \{x^l, x^i\}$$

and note that $\mathcal{D}^{ij} = \mathcal{D}^{ji}$. The metric g will be used to lower indices in analogy with differential geometry. E.g.

$$\mathcal{D}^i{}_j = \mathcal{D}^{ik} g_{kj}, \mathcal{D}_i = g_{ij} \mathcal{D}^j.$$

Moreover, $g = (g_{ij})$ defines a bilinear form on the free module $\mathcal{A}^m$ via

$$g(X, Y) = X^i g_{ij} Y^j, \text{ for } X = X^i e_i \in \mathcal{A}^m \text{ and } Y = Y^i e_i \in \mathcal{A}^m.$$

We introduce the map $\mathcal{D} : \mathcal{A}^m \to \mathcal{A}^m$ by setting

$$\mathcal{D}(X) = \mathcal{D}^i{}_j X^j e_i.$$

Proposition (1.3)

The map $\mathcal{D} : \mathcal{A}^m \to \mathcal{A}^m$ is an orthogonal projection, i.e.

$$\mathcal{D}^2(X) = \mathcal{D}(X) \text{ and } g(\mathcal{D}(X), Y) = g(X, \mathcal{D}(Y))$$

for all $X, Y \in \mathcal{A}^m$. Moreover, $Im(\mathcal{D})$ is isomorphic to $\mathfrak{g}$, which shows that $\mathfrak{g}$ is projective module.

Remark

Note the fact that $\mathfrak{g}$ is a projective nodule is clearly not dependent on whether or not the underlying Poisson algebra has the structure of a Kähler-Poisson algebra, as the definition of $\mathfrak{g}$ involves only inner derivations. Hence, as soon as the Poisson algebra admits the structure of a Kähler-Poisson algebra, it follows that the module of inner derivations is projective. Note that for an arbitrary Lie-Rinehart algebra the module $\mathfrak{g}$ need not to be projective.

Construction of a Kähler-Poisson algebras (1.4)

- Given a Poisson algebra $(\mathcal{A}, \{\cdot, \cdot\})$.
- One may ask if there exist $\{x^1, ..., x^m\}$ and g_{ij} such that $(\mathcal{A}, g, \{x^1, ..., x^m\})$ is a Kähler-Poisson algebras.
- Let us consider the case when $\mathcal{A}$ is a finitely generated algebra, and let $\{x^1, ..., x^m\}$ be an arbitrary set of generators.
- If we denote by $\mathcal{P}$ the matrix with entries $\{x^i, x^j\}$ and by g the matrix with entries g_{ij}, the Kähler-Poisson condition (1) may be written in matrix notation as

$$\eta \mathcal{P} g \mathcal{P} g \mathcal{P} = -\mathcal{P}.$$

- Given an arbitrary antisymmetric matrix $\mathcal{P}$, we shall find g by first writing $\mathcal{P}$ in a block diagonal form, with antisymmetric 2 x 2 matrices on the diagonal.

Proposition (1.5)

Let $\mathcal{P} \in M_N(R)$ be an antisymmetric matrix, and let $\hat{N}$ denote the integer part of $N\,/\,2$. Then there exist $V \in M_N(R)$ and $\lambda_1, ..., \lambda_{\hat{N}} \in R$ such that

$$V^T \mathcal{P} V = diag(\Lambda_1, ..., \Lambda_{\hat{N}}) \text{ if } N \text{ is even}$$
$$V^T \mathcal{P} V = diag(\Lambda_1, ..., \Lambda_{\hat{N}}, 0) \text{ if } N \text{ is odd,}$$

where

$$\Lambda_k = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix}.$$

Returning to the case of a Poisson algebra generated by $x^1, ..., x^m$, assume for the moment that $m = 2N$ for a positive integer N. By Proposition (1.5), there exists a matrix V

$$V^T \mathcal{P} V = \mathcal{P}_0$$

where $\mathcal{P}_0$ is a block diagonal matrix of the form

$$\mathcal{P}_0 = diag(\Lambda_1, ..., \Lambda_N)$$

with

$$\Lambda_k = \begin{pmatrix} 0 & \lambda_k \\ -\lambda_k & 0 \end{pmatrix} .$$

In the same way, defining $g_0 = diag(g_1, ..., g_N)$ with

$$g_k = \frac{\lambda}{\lambda_k} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\lambda = \lambda_1 \cdots \lambda_N$$

we set $g = V g_0 V^T$. As long as det(V) is not a zero divisor, this implies that

$$\mathcal{P} g \mathcal{P} g \mathcal{P} = -\lambda^2 \mathcal{P}.$$

We want to obtain the Kähler Poisson condition

$$\eta \mathcal{P} g \mathcal{P} g \mathcal{P} = -\mathcal{P}.$$

Thus, given a finitely generated Poisson algebra $\mathcal{A}$, the above procedure gives a rather general way to associate a localization $\mathcal{A}[\lambda^{-1}]$ and a metric g to $\mathcal{A}$, such that $(\mathcal{A}[\lambda^{-1}], \{x^1, ..., x^m\}, g)$ is a Kähler Poisson algebra. That is, If λ is invertible, then one obtains a Kähler Poisson algebra with $\eta = \lambda^{-2}$. If λ is not invertible, then one may add the inverse to the algebra to obtain a localization $(\mathcal{A}[\lambda^{-1}]$.

Section (2) The Levi-Civita connection

Since every Kähler Poisson algebra is also a metric Lie-Rinehart algebra, the results of Lie-Rinehart algebras immediately applies. In particular, there exists a unique torsion-free and metric connection on the module $\mathfrak{g}$.

Definition (2.1)

Let $(\mathcal{A}, g, \{x^1, ..., x^m\})$ be a Kähler Poisson algebra and let ∇ be a connection on $\mathfrak{g}$. The connection is called a metric if

$$\alpha(g(\beta,\gamma)) = g(\nabla_\alpha\beta,\gamma) + g(\beta,\nabla_\alpha\gamma)$$

and torsion-free if

$$\nabla_\alpha\beta - \nabla_\beta\alpha - [\alpha,\beta] = 0$$

for all $\alpha, \beta, \gamma \in \mathfrak{g}$.

Proposition (2.2)

If ∇ denote the Levi-Civita connection of a Kähler-Poisson algebra $\mathcal{K}$ then

$$\nabla_{\mathcal{D}^i}\mathcal{D}^j = \frac{1}{2}\mathcal{D}^i(\mathcal{D}^{jk})\mathcal{D}_k - \frac{1}{2}\mathcal{D}^j(\mathcal{D}^{ik})\mathcal{D}_k + \frac{1}{2}\mathcal{D}^k(\mathcal{D}^{ij})\mathcal{D}_k \tag{3}$$

or, equivalently, $\nabla_{\mathcal{D}^i}\mathcal{D}^j = \Gamma^{ij}{}_k\mathcal{D}^k$ *where*

$$\Gamma^{ij}{}_k = \frac{1}{2}\mathcal{D}^i(\mathcal{D}^{jl})\mathcal{D}_{lk} - \frac{1}{2}\mathcal{D}^j(\mathcal{D}^{il})\mathcal{D}_{lk} + \frac{1}{2}\mathcal{D}_k(\mathcal{D}^{ij}). \tag{4}$$

Section (3) Examples

Example (3.1)

Let $\mathcal{A}$ be a unital Poisson algebra generated by two elements $x^1 = x \in \mathcal{A}$ and $x^2 = y \in \mathcal{A}$, and set

$$\mathcal{P} = \begin{pmatrix} 0 & \{x, y\} \\ -\{x, y\} & 0 \end{pmatrix}.$$

It is easy to check that for an arbitrary symmetric matrix g

$$\mathcal{P} g \mathcal{P} g \mathcal{P} = -\{x, y\}^2 det(g) \mathcal{P}.$$

Thus, as long as $\{x, y\}^2 det(g)$ is not zero-divisor, one may localize to obtain a Kähler-Poisson algebra

$$\mathcal{K} = (\mathcal{A}[(\{x, y\}^2 det(g))^{-1}], \{x, y\}, g).$$

Example (3.2)

Let $\mathcal{A}$ be a Poisson algebra generated by three elements $x^1 = x, x^2 = y, x^3 = z \in \mathcal{A}$. Writing $\{x, y\} = a, \{y, z\} = b$ and $\{z, x\} = c$, i.e.

$$\mathcal{P} = \begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{pmatrix}$$

It is easy to check that for an arbitrary symmetric matrix g

$$\mathcal{P}g\mathcal{P}g\mathcal{P} = -\tau\mathcal{P},$$

where

$$\tau = a^2|g|_{33} + b^2|g|_{11} + c^2|g|_{22} + 2ab|g|_{31} - 2ac|g|_{32} - 2bc|g|_{21},$$

and $|g|_{ij}$ denotes the determinant of the matrix obtained from g by deleting the $i'th$ row and the $j'th$ column. Thus one may construct the Kähler-Poisson algebra $\mathcal{K} = (\mathcal{A}[\tau^{-1}], \{x, y, z\}, g)$.

In particular, if g equals the identity matrix, then $\tau = a^2 + b^2 + c^2$.
Let us explicitly work out an example based on an algebra $\mathcal{A}_0$, generated by two elements. Let us start by choosing an element $\lambda \in \mathcal{A}_0$ for which the localization $\mathcal{A} = \mathcal{A}_0[\{x, y\}^{-1}, \lambda^{-1}]$ exists, and then defining the metric as

$$g = \frac{1}{\lambda} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We know that $(\mathcal{A}, \{x, y\}, g)$ is a Kähler-Poisson algebra with $\eta = \frac{\lambda^2}{p^2}$. We introduce $\gamma = \frac{p}{\lambda}$ such that $\eta = \frac{1}{\gamma^2}$ with $p = \{x, y\}$.

Let us start by computing the derivations $\mathcal{D}^x = \mathcal{D}^1$ and $\mathcal{D}^y = \mathcal{D}^2$

$$\mathcal{D}^x = -\frac{1}{\gamma}\{y, .\}$$
$$\mathcal{D}^y = \frac{1}{\gamma}\{x, .\}$$

From Proposition (3.1) one can compute the connection:

$$\nabla_{\mathcal{D}^x}\mathcal{D}^x = \frac{1}{2}\mathcal{D}^i(\lambda)\mathcal{D}_i$$
$$\nabla_{\mathcal{D}^y}\mathcal{D}^y = \nabla_{\mathcal{D}^x}\mathcal{D}^x$$
$$\nabla_{\mathcal{D}^x}\mathcal{D}^y = \mathcal{D}^\lambda$$
$$\nabla_{\mathcal{D}^y}\mathcal{D}^x = -\mathcal{D}^\lambda$$

Moreover, the curvature can be computed

$$R(\mathcal{D}^x, \mathcal{D}^y)\mathcal{D}^x = [\mathcal{D}_x(\lambda)^2 + \mathcal{D}^y(\lambda)^2 - \frac{1}{2}\mathcal{D}_x(\mathcal{D}^x(\lambda)) - \frac{1}{2}\mathcal{D}_y(\mathcal{D}^y(\lambda))]\mathcal{D}^y$$
$$R(\mathcal{D}^x, \mathcal{D}^y)\mathcal{D}^y = [\mathcal{D}_x(\lambda)^2 + \mathcal{D}^y(\lambda)^2 - \frac{1}{2}\mathcal{D}_x(\mathcal{D}^x(\lambda)) - \frac{1}{2}\mathcal{D}_y(\mathcal{D}^y(\lambda))]\mathcal{D}^x$$

Section (4) Summary

- We have introduced the concept of Kähler-Poisson algebras as a mean to study Poisson algebras from a metric of view.
- The relation (1) has consequences that allow for an identification of geometric objects in the algebra, which share properties with their classical counterparts.
- The idea behind the construction was to identify a distinguished set of elements in the algebra that serve as "embedding coordinates", and then use the compatibility as the defining relation for a Kähler-Poisson algebra.

Thank you for your attention